Classical Mechanics Lecture 26

Examples using the ELEs

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Lecture 26 overview

1. Example: a bead sliding on a horizontal rotating rod
2. Example: the simple pendulum via ELEs
3. Constraints in ELEs
    1. Via constraining potential ➔ The tension in the simple pendulum
    2. Via the method of Lagrange multipliers ➔ The tension in the simple pendulum
4. A ball moving down an inclined plane
    1. The ball slipping down the plane (no rolling constraints)
    2. The ball rolling down the plane (with with constraints)

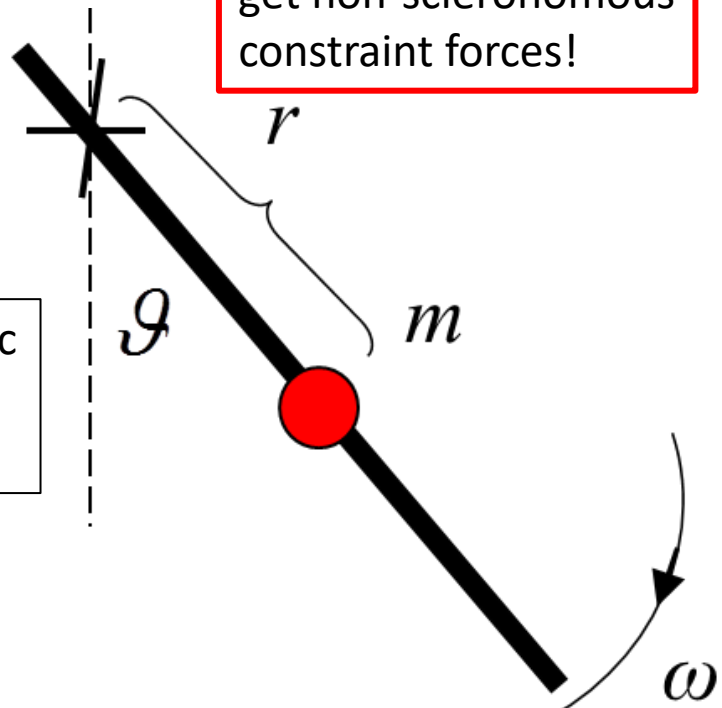
26.1 Example: bead sliding on a driven, rotating, horizontal rod (fixed ω)

- Bead sliding on frictionless rigid rod **driven** at **constant angular speed ω** in the **horizontal** plane ➔ $\theta = \omega t$ ➔ non-scleronomous constraint !
- Using v in plane polar co-ordinates: $\underline{\boldsymbol{v}} = \dot{r}\underline{\hat{\boldsymbol{r}}} + r\dot{\theta}\underline{\hat{\boldsymbol{\theta}}} = \dot{r}\underline{\hat{\boldsymbol{r}}} + r\omega\underline{\hat{\boldsymbol{\theta}}}$
- Lagrangian: $L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 - 0$
- One free coordinate: $q_1 = r$, θ removed via constraint
- The ELE: $\left[\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)\right] = 0$
- Using the above L we get:
- $\frac{\partial L}{\partial r} = mr\omega^2$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$
- Put into ELE & cancel m ➔ $r\omega^2 - \ddot{r} = 0$ is the EoM
- This has exponential solutions: $r = Ae^{\omega t} + Be^{-\omega t}$
- Initial conditions: $r(t=0) = r_0$, $\dot{r}(t=0) = 0$; $A + B = r_0$, $\omega(A - B) = 0$

$$A = B = \frac{r_0}{2}$$
$$r = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})$$
$$r = r_0 \cosh(\omega t)$$

Cannot use Lagrange multiplier method to get non-scleronomous constraint forces!

Similar to harmonic equation but with different sign

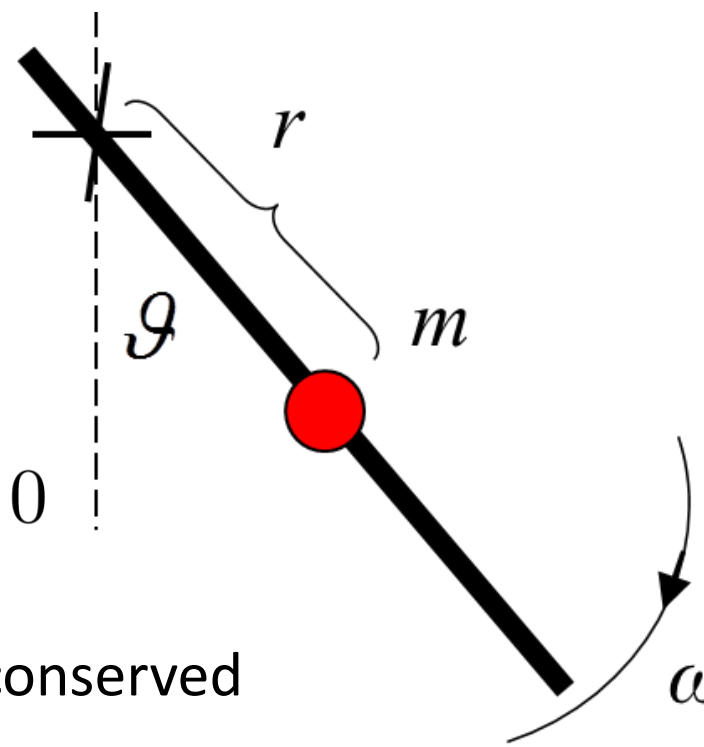


Note: exponentially growing speed and energy. Constraint does works on system

26.1 Example: bead sliding on freely rotating horizontal rod

- Now the rod is not driven but **rotating freely**
- Denote $\omega = \dot{\theta}$ to indicate it is a free variable ➔ $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$
- We get two generalised co-ordinates: $q_1 = r \ , \ q_2 = \theta$
- The two ELEs are:
- As before: $\left[\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)\right] = 0 \Rightarrow r\dot{\theta}^2 - \ddot{r} \overset{[1]}{=} 0$
- L for new variable θ: $\left[\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\right] = 0$
- With: $\underbrace{\frac{\partial L}{\partial \theta} = 0}, \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) \ \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0$

  θ is cyclic variable ➔ conjugate momentum $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = J$ is conserved

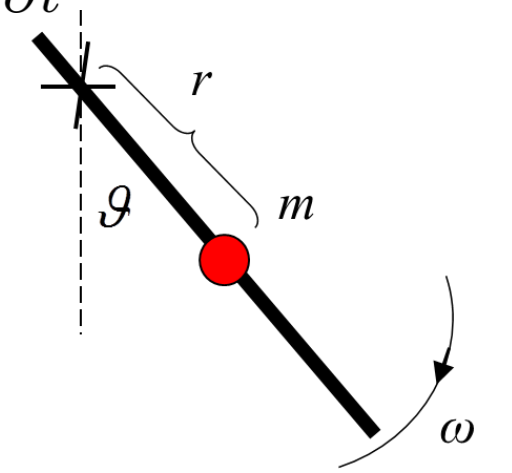


- What does this imply for solution in r ?
- Replace $\dot{\theta} = \frac{J}{mr^2}$ in [1] ➔ $\ddot{r} = \frac{J^2}{m^2r^3}$

This looks hard to solve but …

26.1 Example: bead sliding on freely rotating horizontal rod

- We know that this is a free system and total mechanical Energy is conserved
- Maybe we can use this to solve $\ddot{r} \overset{[1]}{=} \frac{J^2}{m^2 r^3}$
- Use $\frac{J^2}{m^2 r^3} = \frac{1}{2}\frac{\partial}{\partial r}\frac{-J^2}{m^2 r^2}$ and $\ddot{r} = \dot{v}_r$ in [1] ➔ $\dot{v}_r \overset{[2]}{=} \frac{1}{2}\frac{\partial}{\partial r}\frac{-J^2}{m^2 r^2}$
- Multiply [2] with $v_r \Rightarrow v_r \dot{v}_r = \frac{1}{2} v_r \frac{\partial}{\partial r}\frac{-J^2}{m^2 r^2}$ and use $v_r \dot{v}_r = \frac{1}{2}\frac{\partial}{\partial t}\left(v_r^2\right) \Rightarrow$
- $\frac{1}{2}\frac{\partial}{\partial t}\left(v_r^2\right) \overset{[3]}{=} \frac{1}{2} v_r \frac{\partial}{\partial r}\frac{-J^2}{m^2 r^2}$ using $v_r = \frac{\partial r}{\partial t} \Rightarrow v_r \frac{\partial}{\partial r} = \frac{\partial r}{\partial t}\frac{\partial}{\partial r} = \frac{\partial}{\partial t}$
- makes RHS of [3] $\frac{1}{2} v_r \frac{\partial}{\partial r}\frac{-J^2}{m^2 r^2} = \frac{1}{2}\frac{\partial}{\partial t}\frac{-J^2}{m^2 r^2}$ and [3] becomes
- $\frac{1}{2}\frac{\partial}{\partial t}\left(v_r^2 + \frac{J^2}{m^2 r^2}\right) \overset{[4]}{=} 0$ which expresses energy conservation
- Since $E_{trans} = \frac{1}{2} m v_r^2$ & $E_{rot} = \frac{J^2}{2mr^2}$ and now [4] becomes
- $v_r^2 + \frac{J^2}{m^2 r^2} = C \Leftrightarrow \frac{\partial r}{\partial t} = \pm\sqrt{C - \frac{J^2}{m^2 r^2}}$ which separates to



$$\int_{r_i}^{r_f} \frac{\pm \mathrm{d}r}{\sqrt{C - \frac{J^2}{m^2 r^2}}} = \int_{t_i}^{t_f} \mathrm{d}t \Leftrightarrow \frac{\pm 1}{\sqrt{C}} \int_{r_i}^{r_f} \frac{r\,\mathrm{d}r}{\sqrt{r^2 - \frac{J^2}{Cm^2}}} = t_f - t_i \Leftrightarrow \left[\sqrt{r^2 - \frac{J^2}{Cm^2}}\right]_{r_i}^{r_f} = \pm\sqrt{C}\Delta t$$

$$r = \pm\sqrt{ct^2 \pm \sqrt{r_0^2 - \frac{J^2}{m^2}}\,t + r_0^2}$$ where $r_i = r_0$, $r_f = r$, $t_i = 0$, $t_f = t$

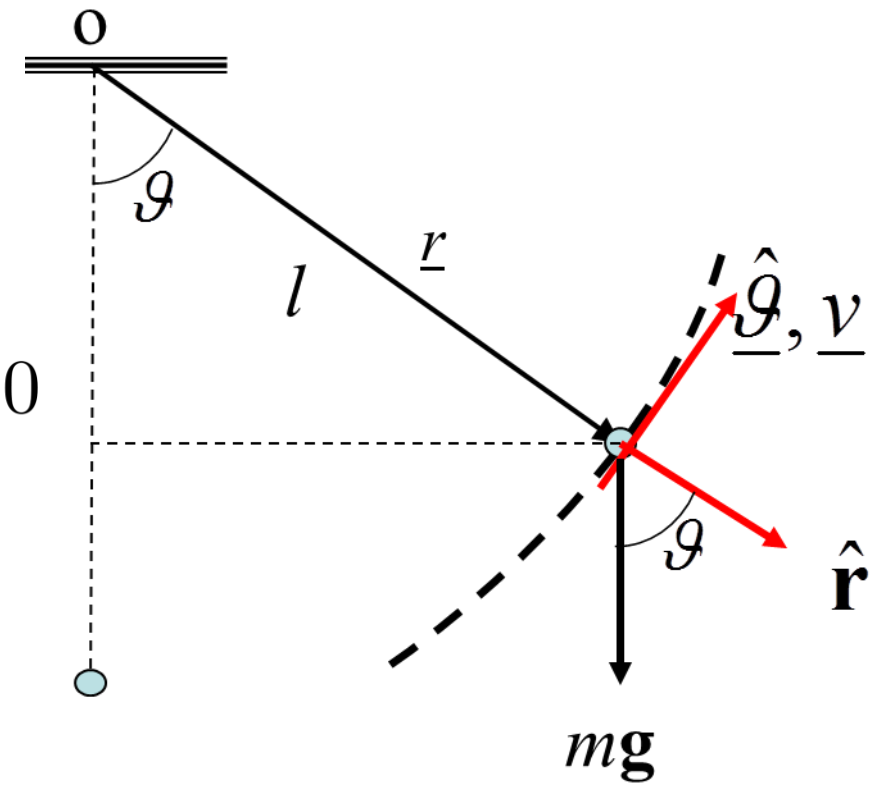
26.2 Example: The simple pendulum via ELE

- Quickly find EoM of simple pendulum via ELE
- Only a single coordinate $q_1 = \theta$ and $v = l\dot{\theta}$
- $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad U = -mgl\cos\theta$

  $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \quad \left[\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)\right] = 0$

  $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}(ml^2\dot{\theta}) = ml^2\ddot{\theta}$

  $\frac{\partial L}{\partial\theta} = -mgl\sin\theta$



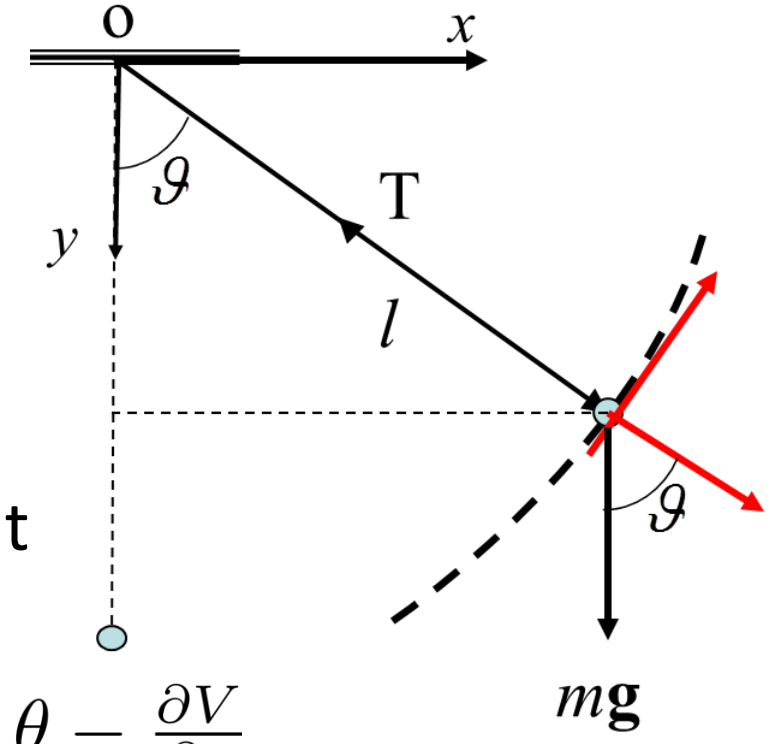
- ELE: $ml^2\ddot{\theta} - mgl\sin\theta = 0 \;\Rightarrow\; \ddot{\theta} + \frac{g}{l}\sin\theta = 0$
- This was fast and gave the expected EoM without using any forces
- But that is also why we did not get the tension in the string from this.
- See next slide how find constraint forces via ELE

26.3.1 Constraints in ELEs via constraining potentials ➔ The tension in the simple pendulum

- The ELE did not say anything about the tension in the string because l was constrained and hence r was not a variable in the ELE. I.e. we never asked about r!
- Initially Re-introduce the radial coordinate (l ➔ r) and add a potential $V(r)$ which constrains the change of r.
- Could use $V(r) = \frac{1}{2}kr^2$ but functional form irrelevant

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta - V(r)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - \frac{\partial V}{\partial r}$$



- Now associate $-\frac{\partial V}{\partial r} = -T$ noting that T points in the $-\hat{\boldsymbol{r}}$ direction.
- Now ELE for r : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Leftrightarrow \quad m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - T$
- Re-introduce the constraint: $r = l \Rightarrow \ddot{r} = 0 \Rightarrow \; 0 = \underbrace{ml\dot{\theta}^2}_{F_{centrip}} + \underbrace{mg\cos\theta}_{\text{Weight}} - \underbrace{T}_{\text{Tension}}$
- Which is what we expect from Newton-II

26.3.2 Constraints in ELEs via Lagrange Multipliers

- The method of Lagrange Multipliers is an **extremely useful**, **widely applicable** (well beyond mechanics) general technique for finding stationary values of functions subject to constraints. We hinted at this in L24, p12.
- Formulate constraint like this: $f(q_i, \ldots, q_n)^{\#} = 0$ (eg. pendulum: $r - l = 0$)
- The original L was: $L(q_i, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)) = T(\dot{q}_1, \ldots, \dot{q}_n) - U(q_i, \ldots, q_n)$
- We create a new Lagrangian as follows:

λ is called a Lagrange multiplier

- $L'(q_i, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \lambda) = T(\dot{q}_1, \ldots, \dot{q}_n) - U(q_i, \ldots, q_n) + \lambda f(q_i, \ldots, q_n)$
- L' is unchanged from L since we only added an arbitrary multiple of zero to L
- We treat the Lagrange multiplier λ as if it was a new coordinate
- Additional ELE for λ replicates constraint eqn. as L independent of $\dot{\lambda}$ by construction. $\left[\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right)\right] = 0$
- Get constraint forces or torques in all co-ordinates via : $Q_i = \lambda \frac{\partial f}{\partial q_i}$

#: Possible but more complex touse this method for non holonomic constraints depending also on the $\dot{q}_i, \ldots, \dot{q}_n$

26.3.2 Constraints in ELEs via Lagrange Multipliers ➔
The tension in the simple pendulum

- Let us apply this method to the simple pendulum case to see if we can derive the tension in the string with: $f(r) = r - l = 0$
- We now have 3 co-ordinates: $q_1 = r \ , \ q_2 = \theta \ , \ q_3 = \lambda$
- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta + \lambda(r-l)$
- r-coord.: $m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta + \lambda$
- θ-coord.: $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta$
- λ-coord.: $0 = r - l$ which just re-generates the constraint equation.
- Now apply the result of the ELE for λ-coord., i.e. set $r{=}l$ for the other coords.
- r-coord.: $0 = ml\dot{\theta}^2 + mg\cos\theta + \lambda \ \Rightarrow \lambda = -ml\dot{\theta}^2 - mg\cos\theta$

  θ-coord. $ml^2\ddot{\theta} + 0 = -mgl\sin\theta$ same EoM as p.8 via potential method
- Now get the constraint forces: $Q_i = \lambda\frac{\partial f}{\partial q_i}$

ELE

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

26.3.2 Constraints in ELEs via Lagrange Multipliers ➔
The tension in the simple pendulum

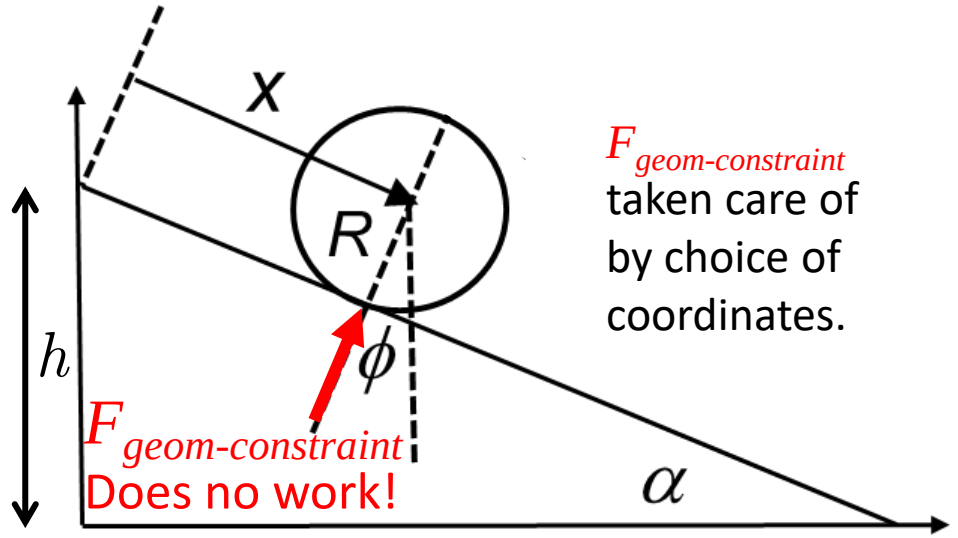
- Get constraint forces: $Q_i = \lambda \frac{\partial f}{\partial q_i} \quad \lambda = -ml\dot{\theta}^2 - mg\cos\theta \quad f(r) = r - l$
- r-coord.: $Q_r = \lambda \frac{\partial f}{\partial r} = (-ml\dot{\theta}^2 - mg\cos\theta)1 = \underbrace{-(m\frac{v^2}{l} + mg\cos\theta)}$ Tension
- θ-coord.: $Q_\theta = \lambda \frac{\partial f}{\partial \theta} = (-ml\dot{\theta}^2 - mg\cos\theta)0 = 0$
- λ-coord.: $Q_\lambda = \lambda \frac{\partial f}{\partial \lambda} = (-ml\dot{\theta}^2 - mg\cos\theta)0 = 0$ But obviously don't need this
- So we got the EoM and the tension without the use of forces in the Ansatz.

26.4.1 ELE for a ball slipping down an inclined plane

- We met this example in L24 as an example of a rolling-constrained systems.
- Set GPE to zero at bottom of the incline:

$$U = mg(h - x\sin\alpha)$$

- Lagrangian: $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - mg(h - x\sin\alpha)$
- Two generalised coordinates: $q_1 = x\,; q_2 = \phi$



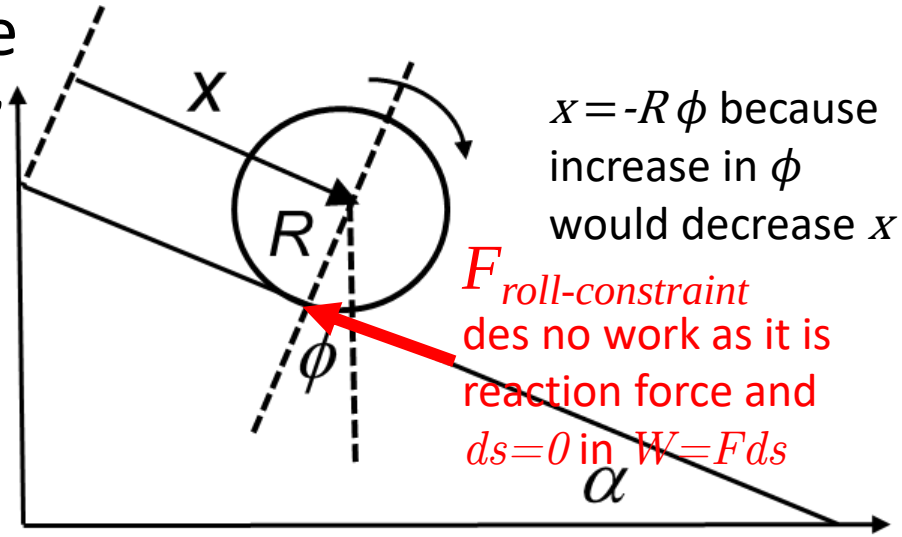
$F_{geom\text{-}constraint}$ taken care of by choice of coordinates.

- **Option 1:** Ball **slipping** (not rolling) ➔ no relation between x and ϕ
  ➔ no rolling-constraint but still geometrical constraint from inclined plain
  - Apply ELE for x: $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \Rightarrow mg\sin\alpha = m\ddot{x} \Rightarrow \ddot{x} = g\sin\alpha$
  - ➔ ball "falls" at angle α wrt. gravity
  - Apply ELE for ϕ: $\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) \Rightarrow 0 = I\ddot{\phi} \Rightarrow$ ϕ is a cyclic coordinate ➔ conjugate momentum p_ϕ is conserved
  - Next impose rolling constraint in two ways

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}$ = Angular momentum of ball is constant

26.5.2 ELE for a ball rolling down an inclined plane

- **Option 2A:** Let the ball **roll** down the rough incline
- Constraint method **A**: "*Direct constraint insertion*"
  - Begin with L of both generalised coordinates $q_1 = x\ ; q_2 = \phi$
  - Apply rolling-constraint $x + R\phi = 0$ directly into the Lagrangian of the unconstrained example.
  - This reduces ELE to one DoF with coordinate x or ϕ
  - ➔ Would loose information about rolling-constraint forces!
  - Since we know the result already, and we will not learn anything about the constraining forces we will not bother to use this method.



26.5.2 ELE for a ball rolling down an inclined plane (continued)

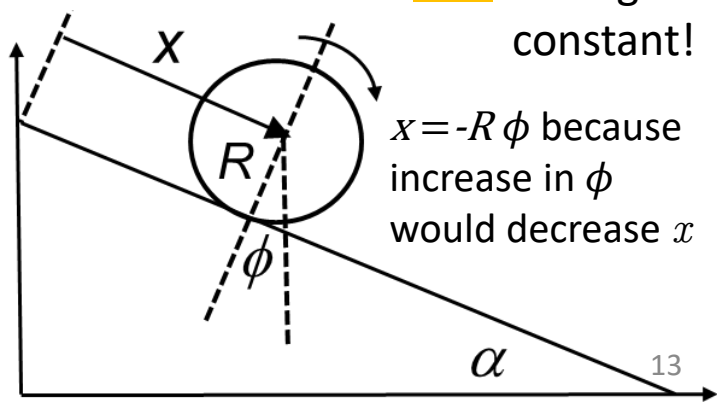
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

- **Option 2B:** The ball **rolling** down the rough incline
- Constraint method **B**: Lagrange Multiplier λ and $L \to L + \lambda f$
  - Constraint function: $f(x,\phi) = x + R\phi = 0$ is also scleronomous and holonomic
  - Lagrangian now has three coordinates: $q_1 = x\,; q_2 = \phi\,, q_3 = \lambda$ ➔ must evaluate 3 ELEs
  - $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - mg(h - x\sin\alpha) + \lambda(x + R\phi)$
  - ELE x ➔ $m\ddot{x} = mg\sin\alpha + \lambda$ [1]
  - ELE ϕ ➔ $I\ddot{\phi} = \lambda R$ [2]
  - ELE λ ➔ $0 = x + R\phi$ [3] = constraint.

To solve these three simultaneous ODEs:
1. get λ from [2] & insert into [1] (x & ϕ become coupled)
$$m\ddot{x} = mg\sin\alpha + \frac{I\ddot{\phi}}{R}$$ [4]
2. from [3] get $\ddot{\phi} = -\frac{\ddot{x}}{R}$ insert into [4] ➔ $\ddot{x} = \frac{g\sin\alpha}{1+\frac{I}{mR^2}}$ [5]

- compare [5] to slipping ball: $\ddot{x}_{roll} = \frac{5}{7}g\sin\alpha < \ddot{x}_{slip} = g\sin\alpha$
- rolling ball has less linear acceleration than slipping ball as energy spent on rotation.
- insert [5b] into [4] ➔ $\ddot{\phi} = -\frac{5g\sin\alpha}{7R}$ [6] also constant and correct sign!
- insert [6] into [2] ➔ $\lambda = -\frac{2}{7}mg\sin\alpha$

$I = \frac{2}{5}mR^2 \Rightarrow \ddot{x} = \frac{5}{7}g\sin\alpha$ [5b] again constant!



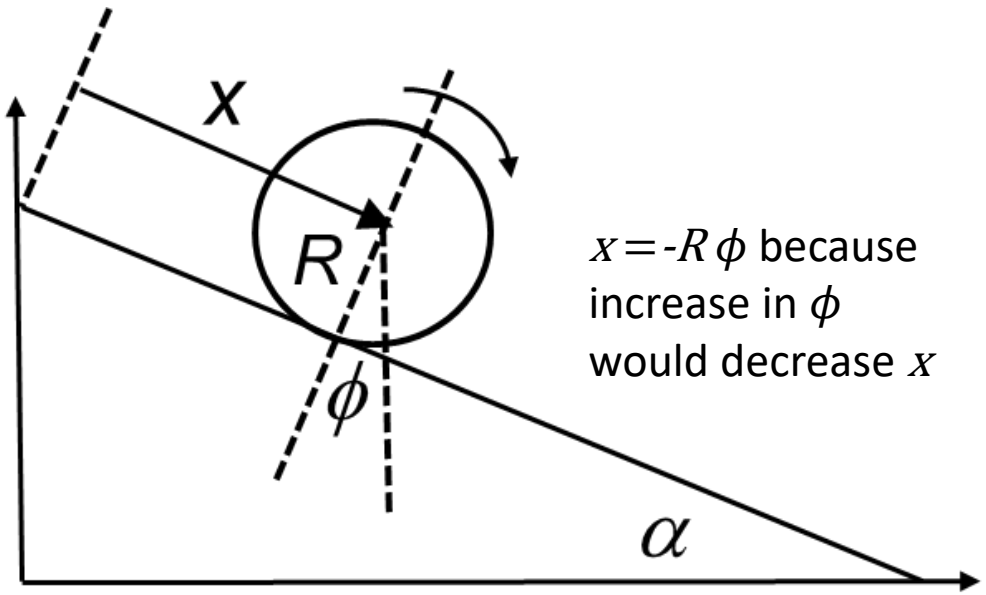
$x = -R\phi$ because increase in ϕ would decrease x

26.5.2 ELE for a ball rolling down an inclined plane (continued)

- **Option 2:** The ball **<u>rolling</u>** down the incline
- Constraint method 2: Lagrange Multiplier λ (continued)
  - Constraint function: $f(x,\phi) = x + R\phi = 0$
  - ELE x ➔ $m\ddot{x} = mg\sin\alpha + \lambda$ [1]
  - ELE ϕ ➔ $I\ddot{\phi} = \lambda R$ [2]
  - ELE λ ➔ $0 = x + R\phi$ [3] = constraint.

$$\left.\begin{aligned} \ddot{x} &= \tfrac{5}{7} g\sin\alpha \\ \ddot{\phi} &= -\frac{5g\sin\alpha}{7R} \\ \lambda &= -\tfrac{2}{7} mg\sin\alpha \end{aligned}\right\}$$

- now get the rolling-constraint forces or torques from: $Q_i = \lambda \frac{\partial f(x_1,\ldots,x_n)}{\partial q_i}$
- $Q_x = F_x = \lambda \frac{\partial(x+R\phi)}{\partial x} = -\frac{2}{7} mg\sin\alpha$
- slows down the "*fall*" relative to slipping case
- $Q_\phi = \tau_\phi = \lambda \frac{\partial(x+R\phi)}{\partial \phi} = -\frac{2}{7} mgR\sin\alpha$
- speeds up the spin via the same force acting at radius R and generating torque
- Above forces are rolling-constraint forces parallel to inclined plain surface. Perpend. forces included by choice of coordinates.



$x = -R\phi$ because increase in ϕ would decrease x