Classical Mechanics Lecture 25

The Euler Lagrange Equations derived via the calculus of variations

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Lecture 25 overview

1. Introduction to the calculus of variations
    1. Example: Fermat's principle and Snell's Law
2. Hamilton's principle of stationary action
3. Conjugate momenta and cyclic variables
4. Summary

From the last lecture

24.4 Simplified equivalence of 1-D Euler-Lagrange Equation to Newton-II

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$ The Euler Langrange Equation

- Motivated 1-D Euler Lagrange Equation extend to generalised coordinates and speeds of an n-particle system so that: $L = L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \quad (k = 1, ..., N_{\mathrm{D}oF})$$

The Euler Langrange Equations

- We will establish the ELE more formally today!

25.1 Introduction to the calculus of variations

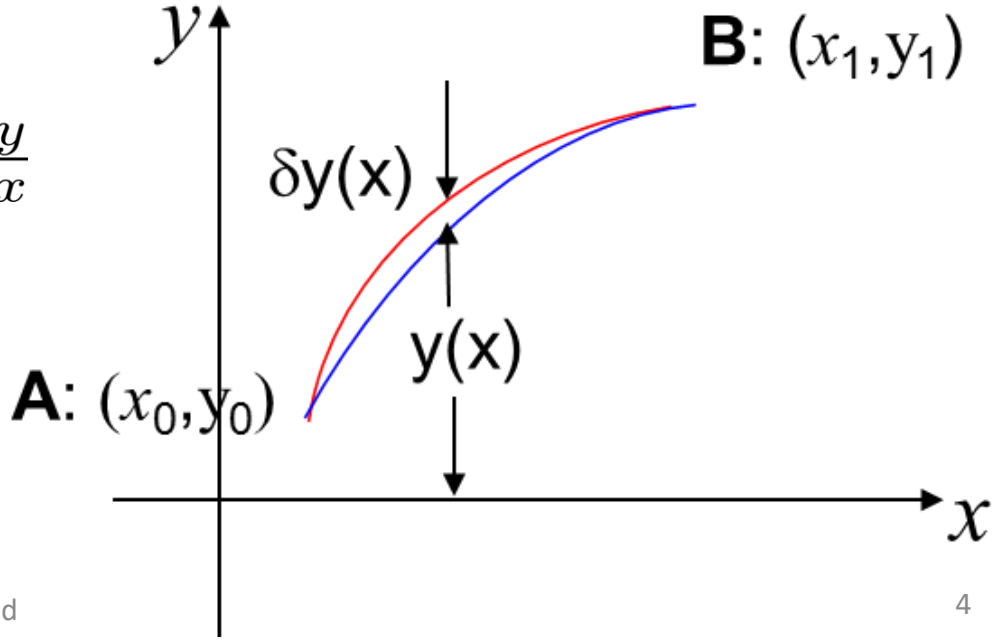
- Take the example of two points in a 2-D plane $A(x_0, y_0)$ and $B(x_1, y_1)$
- Represent paths joining these two points by an equation $y=y(x)$
- All paths must satisfy the boundary conditions (go through A and B):

$$y(x_0) = y_0 \ , \ y(x_1) = y_1$$

- Our task for the calculus of variations is to find **the path** $y(x)$ which results in stationary values of the integral of a chosen scalar function f along $y(x)$.

$$I = \int_{x_0}^{x_1} f(y(x), y'(x))dx$$

- The function f is generally problem-specific
- The function may only depend on $y \ , \ y' = \frac{dy}{dx}$
- We seek a stationary I: $I = \int_{x_0}^{x_1} f(y, y')dx$
- We call I a functional: *"A function of functions"*
- We will find stationary values of I using the calculus of variations.

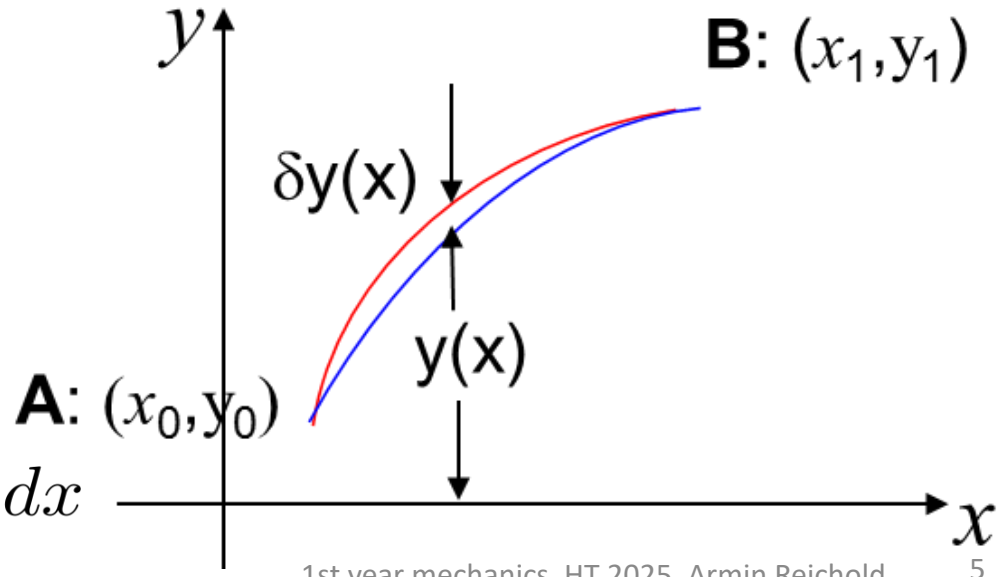


25.1 Introduction to the calculus of variations (continued 2)

- Conditions for the calculus of **variations**:
- Must find functions f that are "representative of our problem"
- To use calculus of variations paths & f should be continuously differentiable
- Use small path variations $\delta y(x)$ to transform paths: $y(x) \rightarrow y(x) + \delta y(x)$
- But keeping the start and end of $y(x)$ unperturbed: $\delta y(x_0) = 0\ ,\ \delta y(x_1) = 0$
- How would the functional's integrand f be affected by this transformation?
- To first order: $f \rightarrow f + \delta f(y, y')$

  $\delta f(y, y') = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \mathcal{O}(\delta y^2, \delta y'^2)$
- where: $\delta y' = \frac{d}{dx}\delta y$
- This makes the variation of our functional:

$$\delta I = \int_{x_0}^{x_1} \delta f(y, y')dx = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}\delta y\right] dx$$



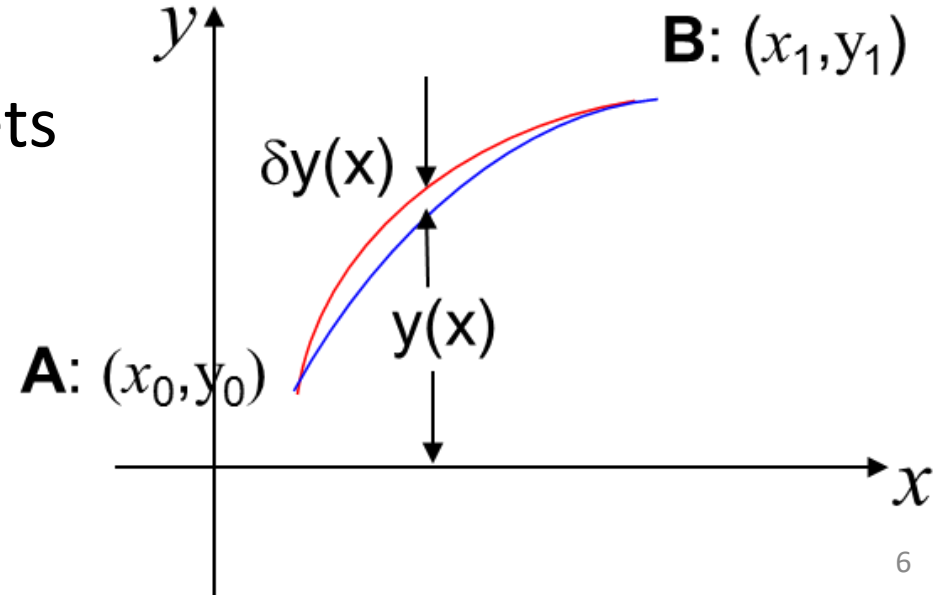
25.1 Introduction to the calculus of variations (continued 3)

- We found: $\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \underbrace{\frac{\partial f}{\partial y'}\frac{d}{dx}\delta y}_{T_2} \right] dx$
- Integrate T_2 by parts: $\int_{x_0}^{x_1} T_2\, dx = \underbrace{\left[\frac{\partial f}{\partial y'}\delta y \right]_{x_0}^{x_1}}_{= 0 \text{ since } \delta y(x_0) = 0\ ,\ \delta y(x_1) = 0} - \int_{x_0}^{x_1} \left(\frac{d}{dx}\frac{\partial f}{\partial y'} \right) \delta y\, dx$
- So: $\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y\, dx$
- For I to be stationary: $\delta I \stackrel{!}{=} 0$ for any δy
- This is only possible if the factor in square brackets vanishes identically

A 1-D Euler Lagrange Equation

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] = 0$$

condition for the functional of f to be stationary



25.1 Introduction to the calculus of variations (continued 1)

- Lets look at a simple example:
- We seek the path $y(x)$ connecting A and B that minimises the path's length between the points
- Task 1: Find the problem-specific function f that represent the path's length via its functional

$$I = \int_{x_0}^{x_1} f(y, y')dx$$

  - must integrate length of line segments dl along our curve $y(x)$: $I = \int_{x_0}^{x_1} dl$

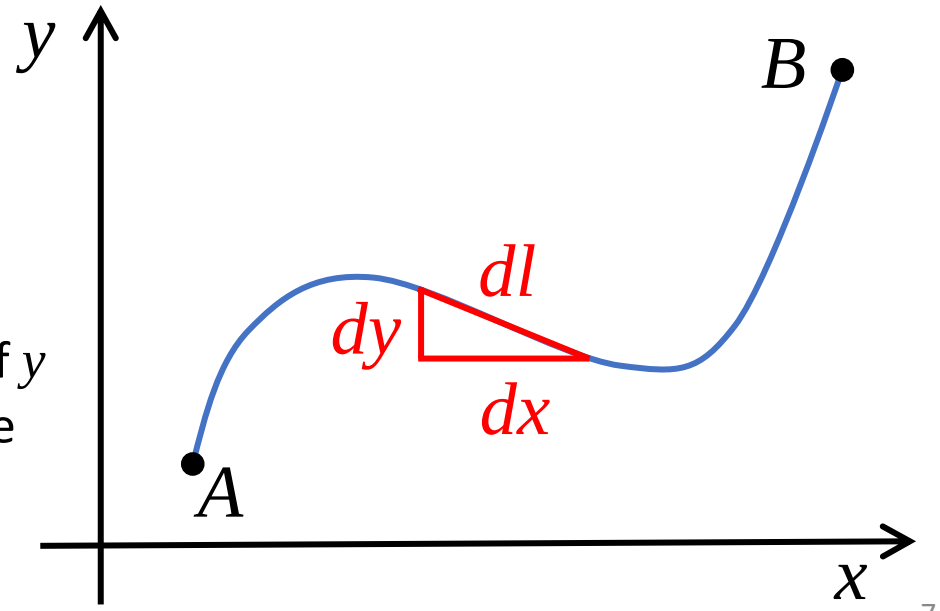
$$dl^2 = dx^2 + dy^2$$

$$dl = \sqrt{dx^2 + dy^2} = \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{f(y')} dx$$

- Task 2: find the $y(x)$ which minimises

$$I = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

f is independent of y as expected for the path length



25.1 Introduction to the calculus of variations (continued 4)

- Let's use the ELE to solve our geometrical, shortest path problem
- Our f was: $f(y') = \sqrt{1+y'^2}$
- calculating terms for the ELE

$$\frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

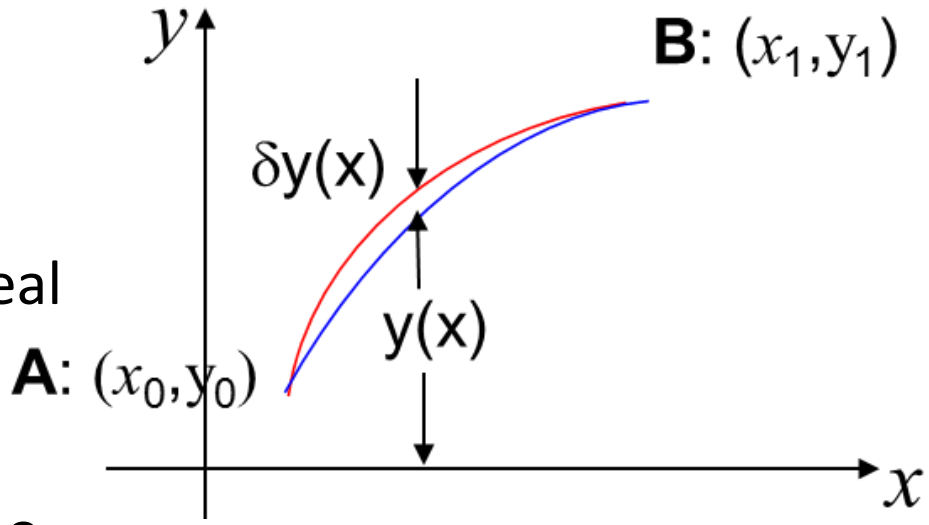
The 1-D Euler Lagrange Equation

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] = 0$$

- Insert this into the ELE $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$ and find y'which satisfies this
- Which means: $\frac{y'}{\sqrt{1+y'^2}} = \text{const} = \alpha$
- Two solutions: $y' = 0 \Rightarrow \alpha = 0$

$$y' = \pm\sqrt{\frac{\alpha^2}{1-\alpha^2}}$$

makes gradient any real number for $0<\alpha<1$

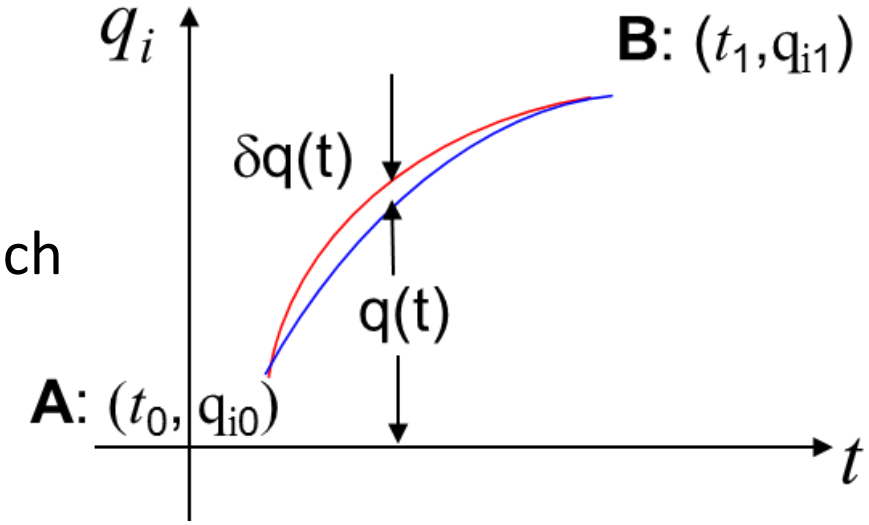


- Proving that the shortest distance between two points is a line of constant slope, i.e. a straight line

$$y(x) = mx + c, m = \pm\sqrt{\frac{\alpha^2}{1-\alpha^2}}$$

25.1 Introduction to the calculus of variations (continued 5)

- So far we have used x (not t) as the path's independent variable and f only depended on a one dimensional path $(y(x), y'(x))$
- The calculus of variations we used to derive the ELE applies equally well to higher dimensional paths.
- In particular it applies to paths in configuration space in which
  - the generalised coordinates are: $(q_1(t), \ldots, q_n(t))$
  - and the generalised speeds are: $(\dot{q}_1(t), \ldots, \dot{q}_n(t))$
  - and the independent variable is t



- So the integral $I = \int_{t_0}^{t_1} f\,(q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t))\,dt$
- has to be stationary wrt. path variations in any number of dimensions $(q_1(t), \ldots, q_n(t))$
- subject to the condition $\delta q_i(t_0) = \delta q_i(t_1) = 0 \;(i = 1, \ldots, n)$
- Works if f satisfies n Euler-Lagrange Equations:
- The ELE tell us how to find the stationary points of the functional $I(f)$

The general Euler Lagrange Equations

$$\left[\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)\right] = 0$$

25.1.1 Example: Fermat's principle and Snell's Law

- A physics example where f and hence the functional is easy to find: Planar refraction
- Find f from Fermat's Principle: "*The path taken by a ray of light between two points A and B is the one that leads to a stationary time of travel.*"
- This f is similar to f for 2-D shortest path, since $dt = \frac{dl}{v}$
- divide f for pathlengths by speed of light in media $v_i = c/n_i$:

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] = 0$$

$$f(y', y) = \frac{\sqrt{1+y'^2}}{c} n(y)$$
$$n(y \geq 0) = n_1$$
$$n(y < 0) = n_2$$

Note: f not differentiable but we recover via physical continuity constraint.

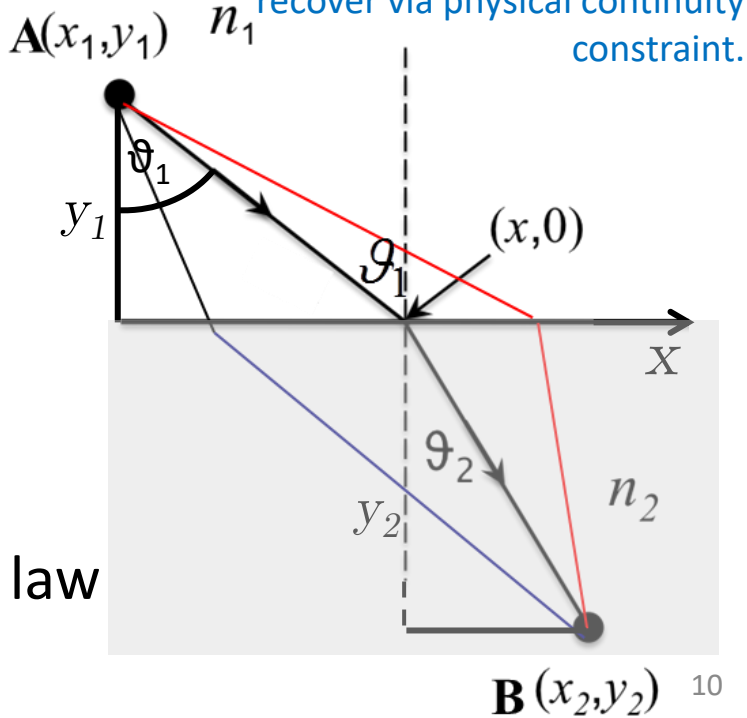
- Already found solution $y(x)$ in regions of constant refractive index to be a straight lines.
- Minimise summed travel time of two straight lines joined at $(x,0)$

$$t(x) = \frac{n_1}{c}\sqrt{(x-x_1)^2 + y_1^2} + \frac{n_2}{c}\sqrt{(x-x_2)^2 + y_2^2}$$

- Minimise wrt. x:

$$\frac{dt}{dx} = \frac{n_1}{c}\overbrace{\frac{x-x_1}{\sqrt{(x-x_1)^2+y_1^2}}}^{\sin\theta_1} + \frac{n_2}{c}\overbrace{\frac{x-x_2}{\sqrt{(x-x_2)^2+y_2^2}}}^{-\sin\theta_2} \stackrel{!}{=} 0$$

$$\frac{n_1 \sin\theta_1}{c} - \frac{n_2 \sin\theta_2}{c} \stackrel{!}{=} 0 \Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2 \text{ Snells's law}$$



25.2 Hamilton's principle of stationary action

- But what replaces Fermat's principle for general problems in mechanics?
- **Hamilton's principle**: Is an independent principle that can be used to determine the EoM of mechanical (and many other) systems. It is absolutely essential to physics!
- In mechanics it is equivalent to Newtons axioms (Newton 1, 2 and 3) or to D'Alembert's principle but we will NOT show either in these lecture (see Hand and Finch, Ch. 1 & 2.3)

Hamilton's Principle

A mechanical system evolves between points A and B in configuration space along that path which makes the **action** functional I **stationary**, meaning:

$\delta I = \delta \int_{t_0}^{t_1} L(q_1(t), ..., q_n(t), \dot{q}_1(t), ..., \dot{q}_n(t))dt = 0$ where L is the Langrangian $L = T - U$

Given Hamilton's principle and the fact that the ELE in general define the paths leading to the stationary values of any associated functional we conclude:

The Euler Lagrange Equations

$$\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] = 0$$

of Mechanics

The ELE of mechanics define the equations of motion of any mechanical system under conservative forces even those derived from time dependent potentials.

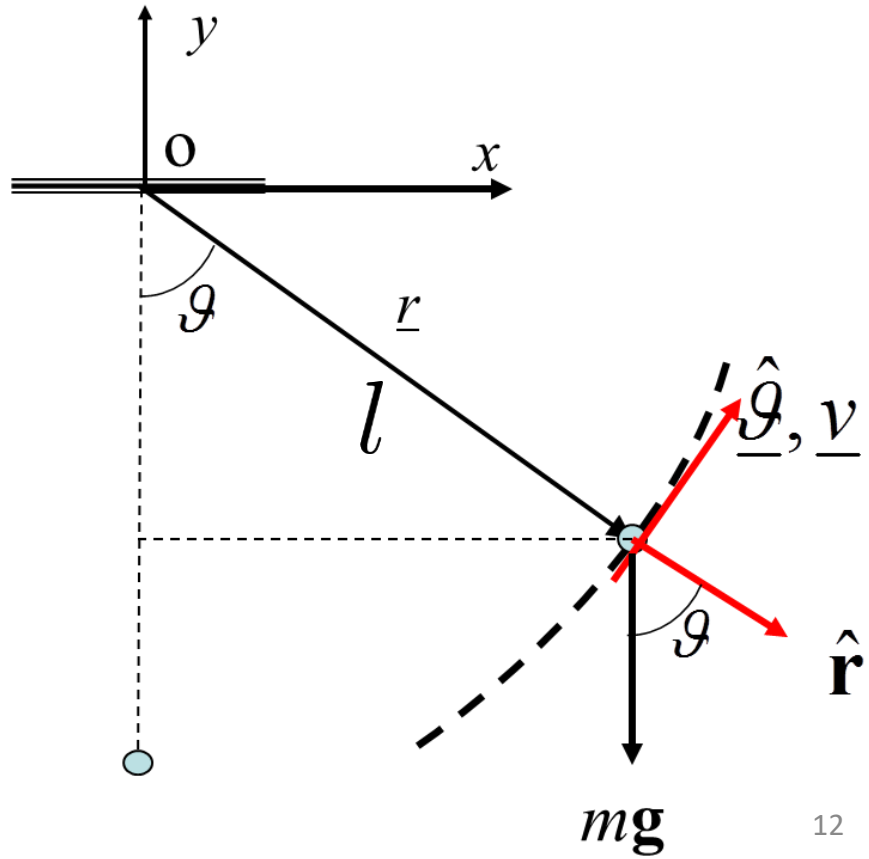
26.1 Conjugate Momenta and cyclic coordinates

- Recall the Euler Lagrange Equations: $\left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] = 0$ where $L = T - U$
- We define the generalised or conjugate momenta: $p_k = \frac{\partial L}{\partial \dot{q}_k}$
- Warning: The p_k are not necessarily linear momenta !
- eg. Consider the simple pendulum:

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(l\dot{\theta})^2 \qquad U = -mgl\cos\theta$$

$$L = \tfrac{1}{2}m(l\dot{\theta})^2 + mgl\cos\theta$$

- We have one DoF and the one co-ord. is θ
- The conjugate momentum to θ is: $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$
- Which is the angular momentum of the pendulum



26.1 Conjugate Momenta and cyclic coordinates

- With this new definition the ELEs read: $\dot{p}_k = \frac{\partial L}{\partial q_k}$ where $p_k = \frac{\partial L}{\partial \dot{q}_k}$
- This is a useful form as it shows clearly that ....
- if L does not explicitly depend on a specific q_k this co-ordinate is called cyclic or ignorable and we get : $\frac{\partial L}{\partial q_k} = \dot{p}_k = 0 \Rightarrow p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{const.}$
- So that this p_k becomes a constant of motion and this is very obvious in this form of the Lagrangian.

The momentum conjugate to a cyclic co-ordinate is a constant of motion

25.3 Summary

- We showed last lecture that the ELE for a simple 1D system give the same EoM as Newtons axioms.
- We showed today, via the calculus of variations, that the n ELE of mechanics express the conditions for the action functional of a system to be minimal.
- We stated that an alternative to Newton's axioms is Hamilton's Principle (the principle of stationary action)
- We already knew that the action is made stationary by applying the ELE.
- We showed that the generalised momentum p_k is conserved if the corresponding coordinate q_k does not explicitly appear in the Lagrangian.
- So far we have only motivated, without proof, that Hamilton's principle will lead in all cases to the same EoMs as Newtons axioms
- Having taken that particular leap of faith (a' la Landau & Lifschitz chapter 1.) or proving the analogy to Newtons axioms via the method's of virtual work (working through chapter 1 of Hand & Finch)
- We can now try to exploit Hamilton's principle expressed via the ELEs and learn how to use them to derive EoM and find constraint forces.