Classical Mechanics Lecture 24

Introduction to Lagrangian Mechanics

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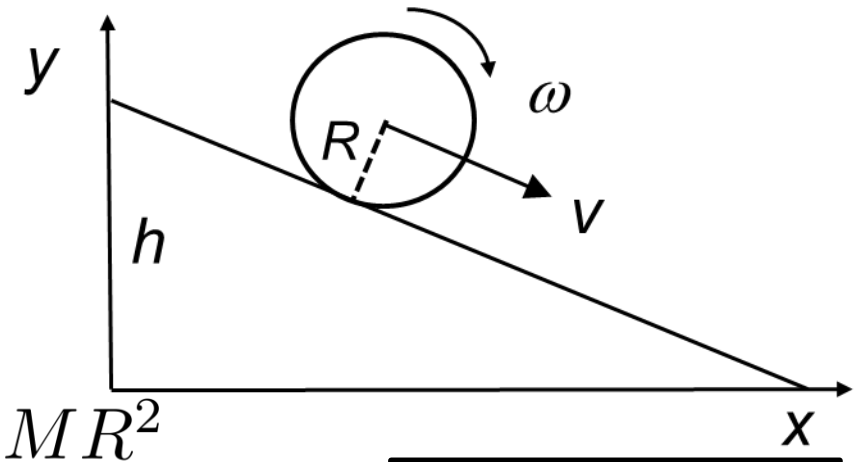
Hilary Term 2024

Errata from L22 and L23

- Thanks to Neil Nair for noticing that L22, slide 12 contained a Freudian slip revealing a desire for summer. The word summerty should really have been symmetry!
- Thanks to Louisa Macaire for noticing in L23, slide 8 that I really butchered the comparision between a cylinder and sphere rolling down an incline. I used the wrong MoI for the cylinder. The slide is repeated on the next page

23.3 Example 1: solid ball rolling down a slope

- Ball starts at rest, height $y=h$, in CM frame rotates around principle axis: find v_f
- [Total Energy of ball] = [Rotational KE in CM] + [KE of CM] + [gravitational PE]
- $E = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 + Mgy$
  where we chose potential energy to be zero at $y=0$
- Using Energy conservation:
  $E_i = E_f \Leftrightarrow Mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_f^2$
- For a solid sphere we found in L22, p13: $I_{\text{sphere}} = \frac{2}{5}MR^2$
- $2gh = Mv_f^2\left(\frac{I}{MR^2}+1\right) \quad \Rightarrow v_f = \sqrt{\frac{2gh}{\frac{I}{MR^2}+1}}$
- ➔ $\frac{I_{\text{shere}}}{MR^2} = \frac{2}{5} \Rightarrow v_{f,\text{sphere}} = \sqrt{\frac{10}{7}gh}$



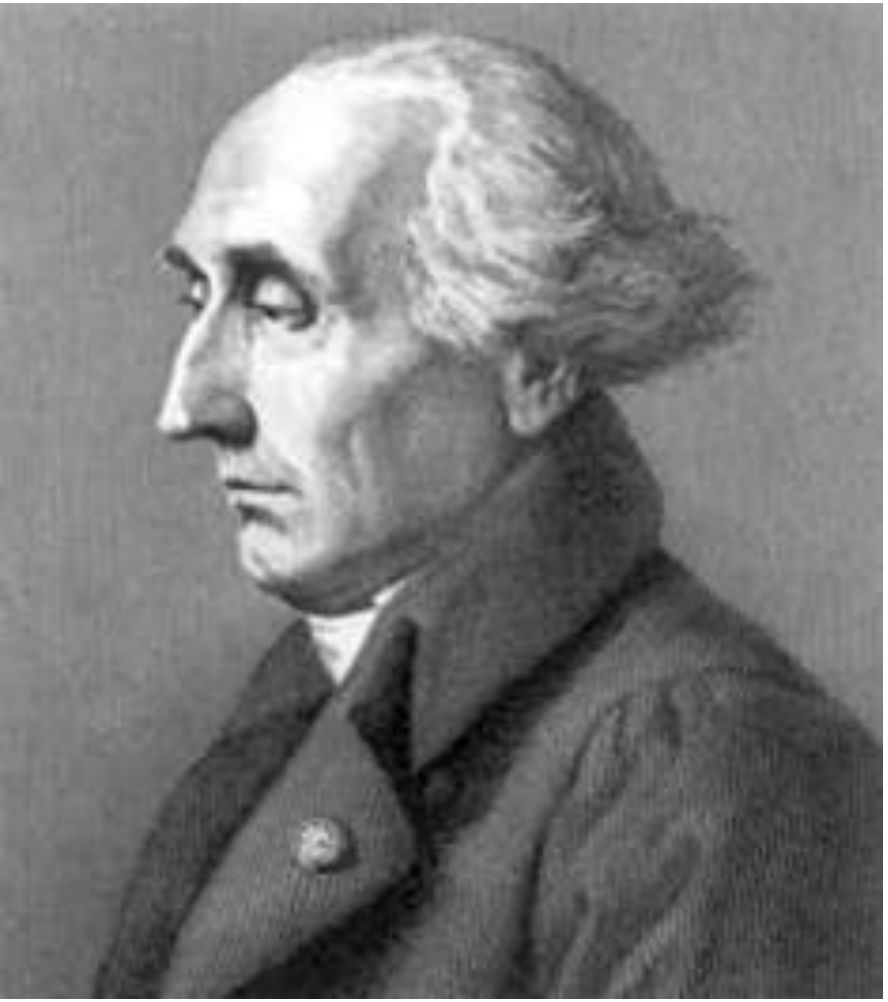
ω and v coupled since ball is rolling and not sliding!
$\omega = v/R$

- Compare to solid cylinder: $\frac{I_{\text{cyl}}}{MR^2} = \frac{1}{2} > \frac{I_{\text{spere}}}{MR^2} \Rightarrow v_{f,\text{cyl}} = \sqrt{\frac{4}{3}gh} < v_{f,\text{sphere}}$
- Cylinder gets to the bottom after the sphere of same mass and same radius due to the cylinder's larger MoI

Lecture 24 overview

1. Introduction to Lagrangian Mechanics
2. The energy method to obtain the Equation of Motion
3. Familiarising with the jargon
    1. Generalised co-ordinates
    2. Degrees of freedom
    3. Configuration Space
    4. Constraints
4. Simplified derivation of the Euler-Lagrange Equations via Newton-II

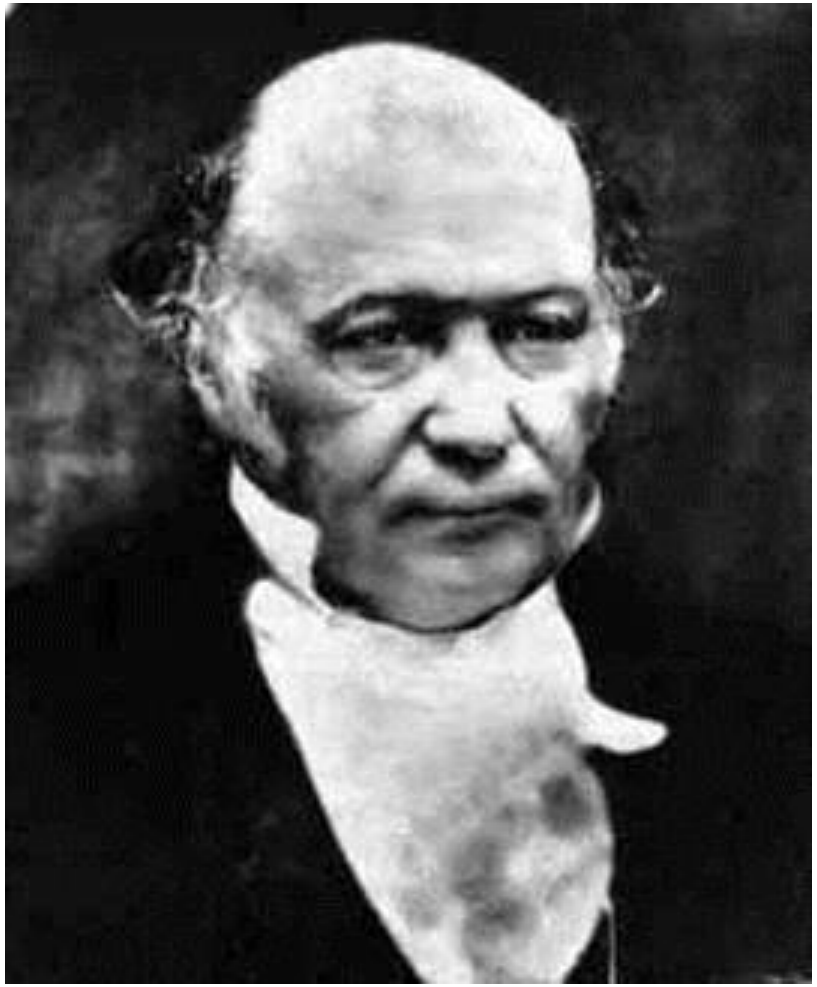
24.1 A new set of “heroes”



Joseph-Louis Lagrange (1739-1810)



Emmy Noether (1882-1935)



Sir William Rowan Hamilton (1805-1865)

24.1 Introduction to Lagrangian Mechanics

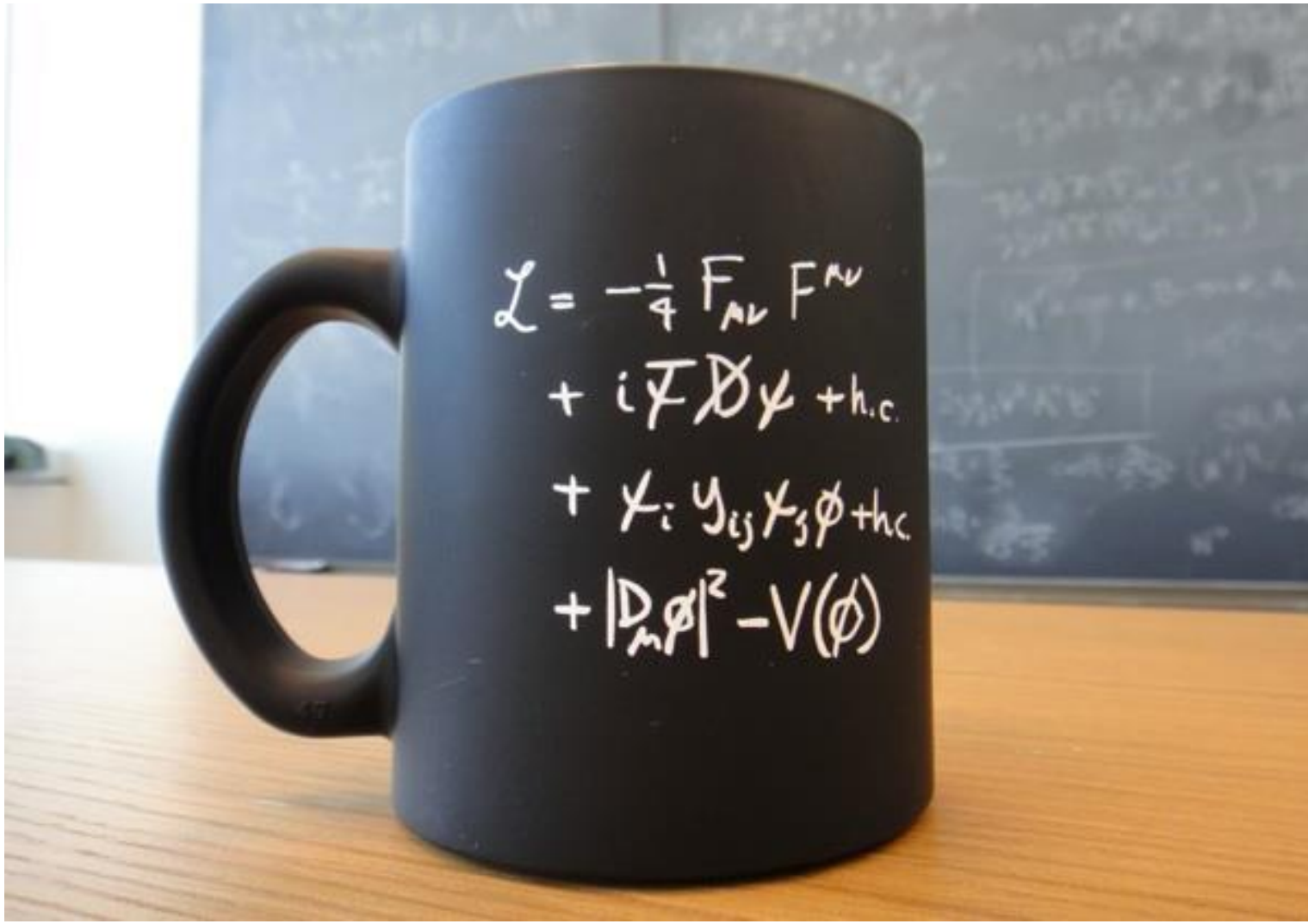
- What Mechanics?
  - Theoretical Mechanics: A method to get from a set of axioms to the EoM. & methods to solve them
  - Experimental Mechanics: To apply and compare the solutions of the EoM to experimental results
- Many such axioms exist:
  - Newton's three axioms
  - d'ALamberts's principle
  - principle of least action
  - the principles of energy and momentum conservation
  - maybe even Mach's principle
- Newtonian mechanics:
  - Newton's 3 axioms & many force vectors ➔ EoM = one vector valued 2^{nd} order ODE per point particle
- Lagrangian mech.:
  - Principle of least action & scalar Lagrangian ➔ EoM = coupled scalar 1^{st} order PDEs. One PDE per degree of freedom. These include an additional ordinary time derivative.

24.1 Introduction to Lagrangian Mechanics

- Newton's laws relate many force vectors to acceleration vectors
- Lagrangian mechanics avoids the concept of force:
  For complicated situations, it may be hard to identify all the forces, especially if there are constraints on the motion as in complex engineering mechanics.
- The Lagrangian is a single scalar function of the system's co-ordinates and velocities.
- The Lagrangian treatment provides:
  - framework to generate conservation laws from continuous symmetries of the Lagrangian (Noether's theorem)
  - Method to incorporate constraints of motion via Lagrange multipliers
  - Method to obtain the constraint forces
- Lagrangian methods appear in most areas of fundamental physics
  - special and general relativity,
  - electromagnetism,
  - quantum mechanics,
  - quantum field theory, ...

24.1 Introduction to Lagrangian Mechanics continued

- Example Lagrangian: all we know about
  - All known elementary particles
  - three of the four fundamental interactions
  - the way to give particles a mass
- can be expressed concisely on a tea cup via the Standard Model Lagrangian.



24.1 Introduction to Lagrangian Mechanics continued

- But one could expand the level of detail in the standard model Lagrangian

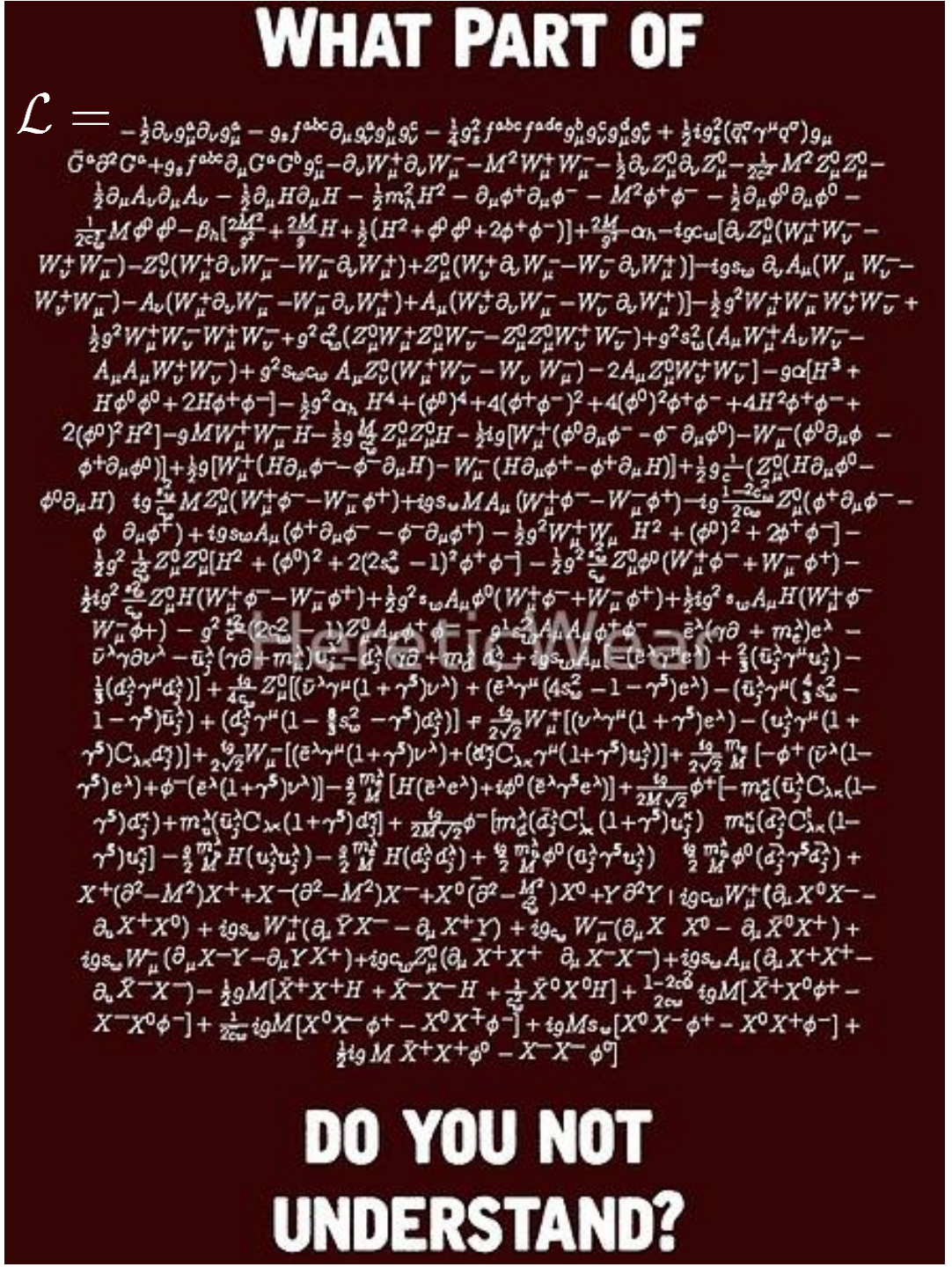
$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) - \frac{1}{2}tr(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}) && \text{(U(1), SU(2) and SU(3) gauge terms)} \\
& + (\bar{\nu}_L, \bar{e}_L)\bar{\sigma}^\mu i D_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{e}_R \sigma^\mu i D_\mu e_R + \bar{\nu}_R \sigma^\mu i D_\mu \nu_R + (\text{h.c.}) && \text{(lepton dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[(\bar{\nu}_L, \bar{e}_L)\phi M^e e_R + \bar{e}_R \bar{M}^e \bar{\phi}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}\right] && \text{(electron, muon, tauon mass term)} \\
& - \frac{\sqrt{2}}{v}\left[(-\bar{e}_L, \bar{\nu}_L)\phi^* M^\nu \nu_R + \bar{\nu}_R \bar{M}^\nu \phi^T \begin{pmatrix} -e_L \\ \nu_L \end{pmatrix}\right] && \text{(neutrino mass term)} \\
& + (\bar{u}_L, \bar{d}_L)\bar{\sigma}^\mu i D_\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R \sigma^\mu i D_\mu u_R + \bar{d}_R \sigma^\mu i D_\mu d_R + (\text{h.c.}) && \text{(quark dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[(\bar{u}_L, \bar{d}_L)\phi M^d d_R + \bar{d}_R \bar{M}^d \bar{\phi}\begin{pmatrix} u_L \\ d_L \end{pmatrix}\right] && \text{(down, strange, bottom mass term)} \\
& - \frac{\sqrt{2}}{v}\left[(-\bar{d}_L, \bar{u}_L)\phi^* M^u u_R + \bar{u}_R \bar{M}^u \phi^T \begin{pmatrix} -d_L \\ u_L \end{pmatrix}\right] && \text{(up, charmed, top mass term)} \\
& + \overline{(D_\mu \phi)} D^\mu \phi - m_h^2[\bar{\phi}\phi - v^2/2]^2/2v^2. && \text{(Higgs dynamical and mass term)} \quad (1)
\end{aligned}
$$

24.1 Introduction to Lagrangian Mechanics continued

- and expanded further ....
- If you must buy this T-shirt please add:

$$\mathcal{L} =$$

to the top left corner



24.2 The energy method to obtain the Equation of Motion

- Consider a single particle moving under the influence of a conservative force in one dimension (i.e. mass on a spring but moving on a linear rail)
- Energy of this system: $E = \frac{1}{2}m\dot{x}^2 + U(x)$ is a conserved quantity ➔

  $\dot{E} = m\dot{x}\ddot{x} + \frac{\partial U}{\partial x}\dot{x} = 0 \Leftrightarrow \dot{x}(m\ddot{x} + \frac{\partial U}{\partial x}) = 0$ [1]
- From definition of a conservative force: $F = -\frac{\partial U}{\partial x}$
- Insert F into LHS of [1]: $\dot{x}(m\ddot{x} - F) = 0$
- The formal solutions of this are:

  Trivial: $\dot{x} = 0$ Newton-II: $F = m\ddot{x}$ Newton 0: $\dot{x} = \text{const}\ \&\ F = 0 \Rightarrow \ddot{x} = 0$
- Example: 1D mass on a linear spring: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$
- $\dot{E} = 0 \Rightarrow m\dot{x}\ddot{x} + kx\dot{x} = 0 \Rightarrow m\ddot{x} + kx = 0$ which is the harmonic eqn.
- Never said the word force but got EoM directly from energy conservation

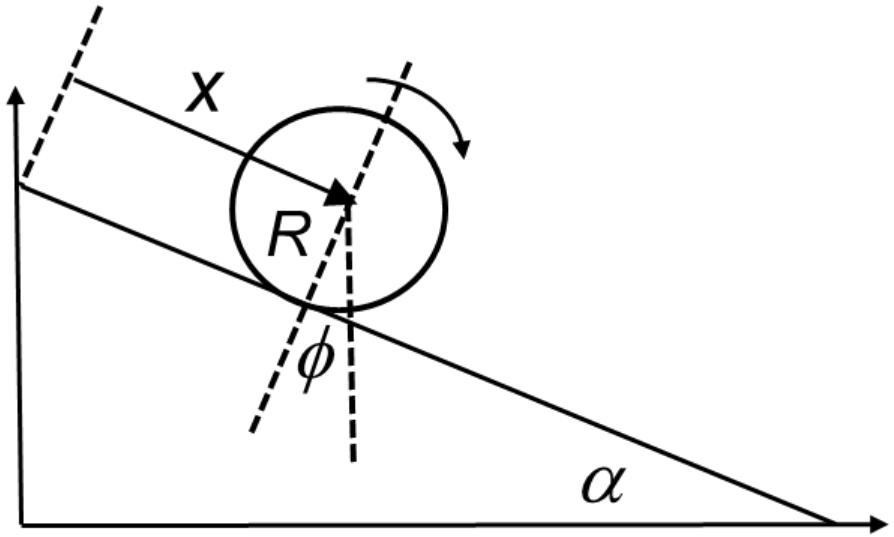
24.3 Becoming familiar with the jargon

- Jargon is important! Maybe call it nomenclature to sounds more important ?
- Generalised coordinates:
  - Any set of parameters $q_k(t)$ that completely specifies the system's configuration. q_k are typically geometrical parameters such as co-ordinates x, y, z or angles etc. Nothing to be worried about.

- Generalised velocities:
  - Simply the time derivatives of the generalised coordinates: $\dot{q}_k(t)$

- Degrees of Freedom (DoF)
  - The number of degrees of freedom (NDoF) is **THE** number of independently variable generalised coordinates which is sufficient to uniquely describe the configuration of the system

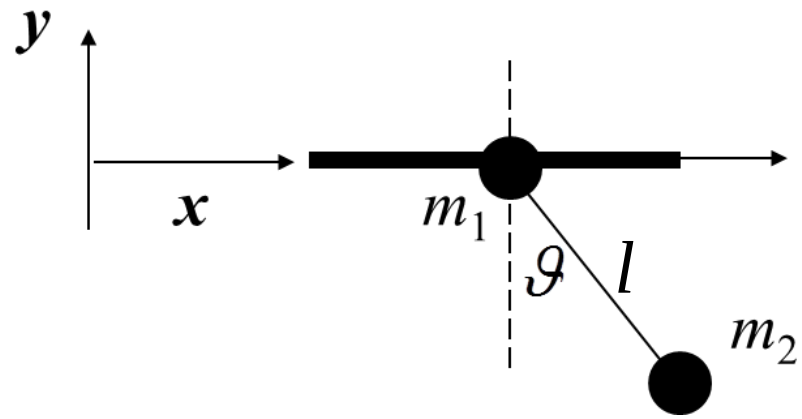
24.3.2 Becoming familiar with the jargon: Examples of Degrees of freedom



- Ball rolling down an incline

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\phi}^2 - mgx\sin\alpha$$

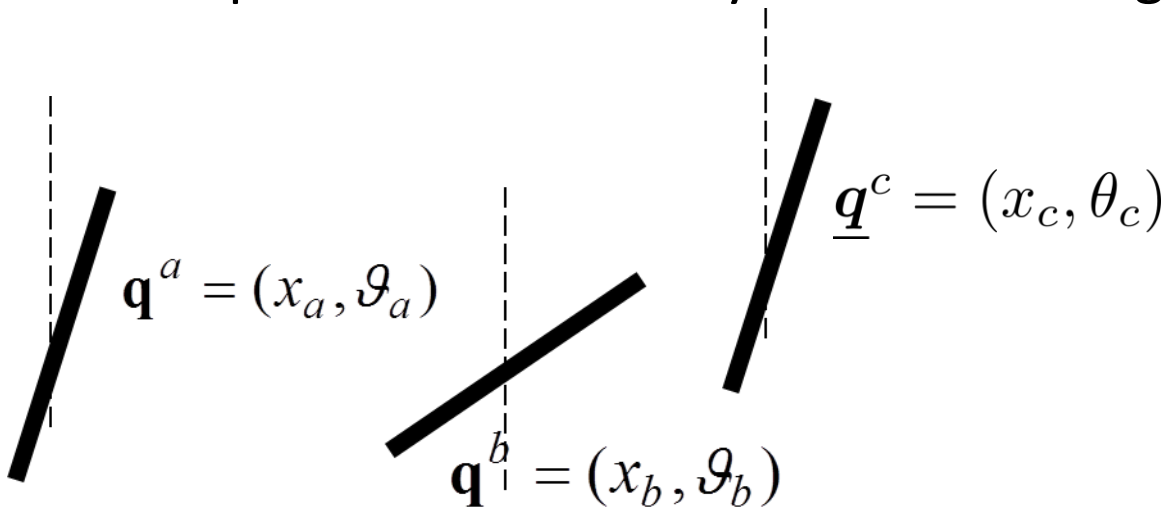
- But x and ϕ are coupled: $x = R\phi \Rightarrow \dot{x} = R\dot{\phi}$
- The problem is reduce to one generalised coordinate and velocity: $q_1 = x\ ; \dot{q}_1 = \dot{x}$
- Originally a 2D problem (x,y) or (x, ϕ) but
- sloped surface introduced a constraint ➔
- This system has one degree of freedom

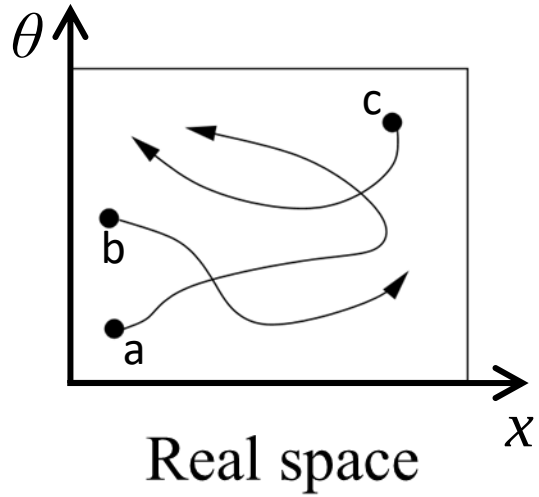


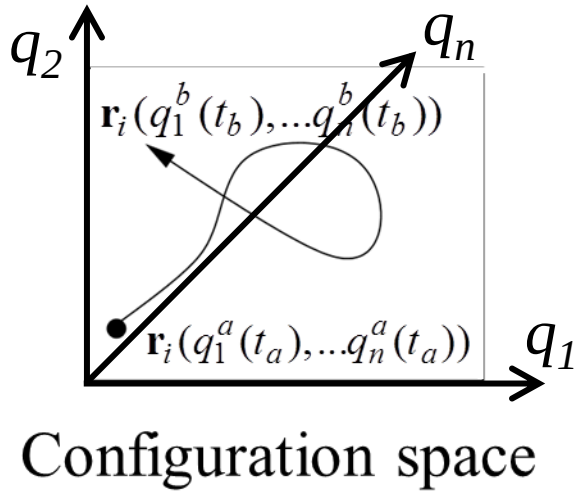
- Pendulum of mass m_2 with pivot of mass m_1 freely moving in x direction
- Pivot coordinates: *(x,0)*
- Pendulum coordinates: $(x + l\sin\theta,\ -l\cos\theta)$
- This system has two degrees of freedom: x and θ

24.3.4 Becoming familiar with the jargon: Configuration Space

- The configuration space of a system with n DoF is an n-dimensional space whose points determine the spatial positions of the system in time. The coordinates in this space are the generalised coordinates $\underline{\boldsymbol{q}} = (q_1, \ldots, q_n)$
- Example 1: Configuration space co-ordinates of one free point mass in 3D space can be any set of Euclidian coordinates such as $\underline{\boldsymbol{q}} = (x, y, z)$ or $\underline{\boldsymbol{q}} = (r, \theta, \phi)$
- Example 2: Three rods with their CoM moving in 1D at locations x_a, x_b , x_c and angles θ_a, θ_b , θ_c have configuration space coordinates: $\underline{\boldsymbol{q}} = (x_a, x_b, x_c, \theta_a, \theta_b, \theta_c)$
- The path of the entire system is one single continuous trajectory in configuration space







24.3.3 Becoming familiar with the jargon: Constraints

- A system has constraints if its components are not permitted to move freely in 3-D, eg.:
  - A particle moving on the plane of a table is restricted to move on a 2-D plane
  - A mass on a simple 2D-pendulum is restricted to oscillate at a fixed length l from a pivot and it moves on a 1-D line (a circular arc).
- Constraints and the constrained system with $\boldsymbol{n}$ objects are "*Holonomic and Scleronomous*" if:
  - constraint is independent of time (i.e. scleronomous) eg. plane of table fixed in time
  - and if these scleronomous constraints can further be written as a mix of the following forms:
  - $f_k(q_1, ..., q_{3n}) = 0 \quad (k = 1, 2, ..., r \leq 3n)$:

    r eqns. of ($3n$-1)-dimensional hyper surfaces, in a $3n$-dimensional configuration space or
  - $f_{k,1}(q_1, ..., q_{3n}) = 0$, $f_{k,2}(q_1, ..., q_{3n}) = 0$ $(k = 1, 2, ..., r \leq \frac{3}{2}n)$:

    r pairs of eqns. of ($3n$-2)-dimensional hyper-curves in $3n$-dimensional configuration space
  - then constraints describe geometrical shapes to which movements of particles are constrained.
  - Using only co-ordinates and no velocities in constraints guarantees that DoF reduce from initial N_{DoF}= $3n$ (unconstraint) to N_{DoF}=$3n$-r . This reduction in DoF makes constraints holonomic.
- We will only work with holonomic constraints!!
  3D-Rolling constraint for sphere: $\underline{\boldsymbol{v}} - R\underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{n}} = 0$ is non-holonomic. Cylinder rolling is holonomic.

24.3.3 Becoming familiar with the jargon: Constraints continued

- Adding constraints to a system via these equation may sound un-physical as we expect all influences on the motion of particles to be described by forces.
- We could express the constraints via so called "*constraint forces*" or "*reaction forces*" F' in contrast to the external forces we considered hitherto.
- Using constraint forces the EoM would look like those of free bodies:

$$m_i\ddot{r}_i = F_i + F_i' \quad (i = 1, 2, ..., n)$$

- If we assume scleronomous constraint forces to only act perpendicular to the trajectories of our particles they never do any work on our bodies. Note: Friction is not a constraint force although it is a reaction force!
- Note: The above assumption sounds very plausible indeed but it does not follow from any axioms and hence remains an assumption
- This perpendicularity will allows us to obtain the F' from the constraint equations f via:
- $F_{i,k}' = \lambda \frac{d}{dq_i}(f_k) = \lambda \nabla f_k \quad (k = 1, 2, ..., r \leq 3n)$ λ is an arbitrary *Lagrange Multiplier*
- We will meet these *Lagrange Multipliers* again in Lecture 26 to derive constraint forces!
- Because constraint forces do no work and we can get them from constraint equations we may describe a systems by external forces and constraint equations without loss of information.

24.4 Simplified derivation of the Euler-Lagrange Equations via Newton-II

- Consider the Lagrangian: $L = T - U$
  where T is total kinetic and U total potential energy of the system.
- 1-D Example:
- $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$ has no dependence on x
- $U(x) =$ any function of x without dependence on $\dot{x}$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$$

- How can we find the EoM from L?
- Partially differentiate L wrt. x : $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_{cons}$ [1]
- Partially differentiate L wrt. $\dot{x}$: $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$ [2]
- Differentiate [2] wrt. time: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} = F$ and equate with [1]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$ The Euler Langrange Equation

24.4 Simplified derivation of the Euler-Lagrange Equations via Newton-II

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$ The 1-D Euler Langrange Equation (ELE)

- Next generalise Euler Lagrange Equation to use generalised coordinates and speeds of an n-particle system so that: $L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$
- Because the ELE involves only partial differentiation we can write down one ELE for every q_k

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \quad (k = 1, \ldots, N_{\mathrm{DoF}})$$

The Euler Langrange Equations

- We will do this more formally in the next lecture!