Classical Mechanics Lecture 20

Open orbits

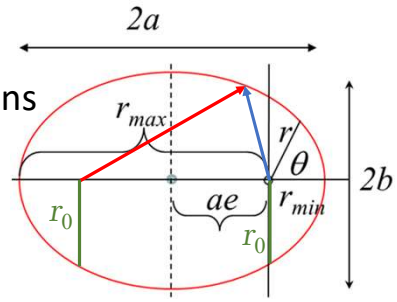
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Hilary Term 2025

Lecture 20 overview

1. Open hyperbolic orbits
2. Distance of closest approach in a hyperbolic orbit
3. The angle of deflection in a hyperbolic orbit
    1. Method 1: using impulse
    2. Method 2: using hyperbolic orbit parameters
    3. Rutherford scattering onto a hyperbolic orbit

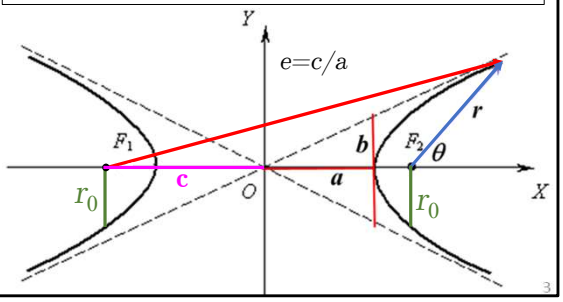
20.1 Open Hyperbolic Orbits

- Recall the orbit equation (L18, p8):
  $r(\theta) = \frac{r_0}{1+e\cos\theta}$ describes all conic sections
- For an ellipse we had $0<e<1$ leading to:
- Cartesian form: $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$
- with a and b as shown to the right
- For the Hyperbola we require $e>1$ leading to:
- cartesian form: $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$
- Why is this so? Do the geometry or …
- Show plausibility by looking at asymptotes
  $\lim_{x\to\infty, y\to\infty}(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1 \Rightarrow (\frac{x}{a})^2 = (\frac{y}{b})^2$
  $\frac{x}{a} = \pm\frac{y}{b} \qquad y = \pm\frac{b}{a}x$
- Which are the asymptotes of the hyperbola



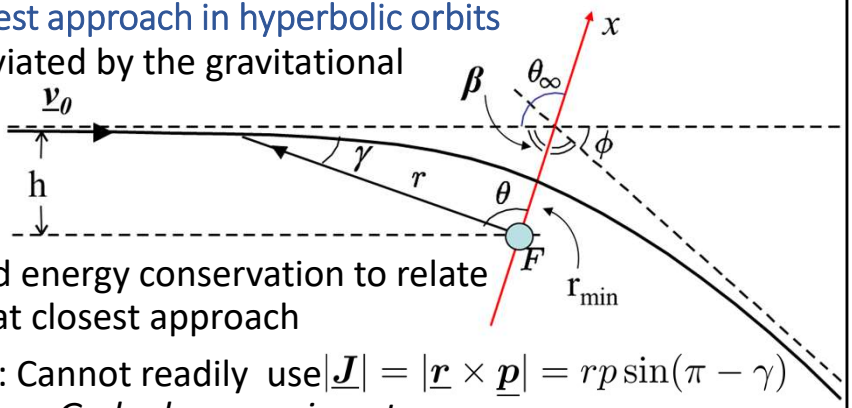
Euclidian definition:
A ellipse is the set of all points (x,y) in a plane such that the sum of the distances between (x,y) and the foci is a positive constant.

A hyperbola is the set of all points (x,y) in a plane such that the difference of the distances between (x,y) and the foci is a positive constant (2a).



20.2 Distance of closest approach in hyperbolic orbits

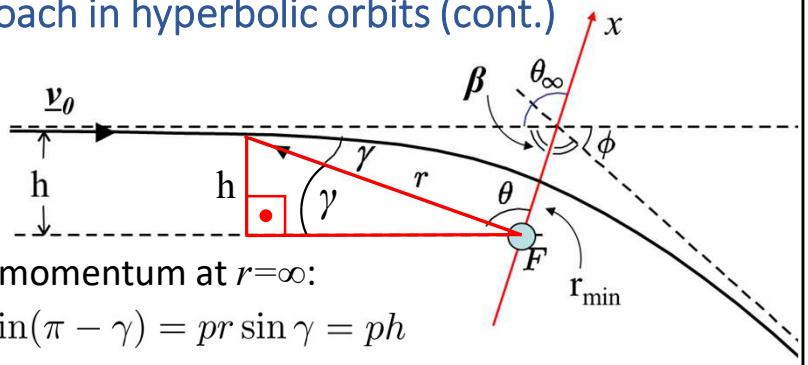
- Example of Oumuamua, deviated by the gravitational attraction of the sun
- At $r=\infty$ we have $\underline{v} = \underline{v}_0$
- h is the impact parameter



- Use angular momentum and energy conservation to relate properties at $r=\infty$ to those at closest approach
- Angular momentum at $r=\infty$: Cannot readily use $|\underline{J}| = |\underline{r} \times \underline{p}| = rp\sin(\pi - \gamma)$ since $r=\infty$ and $\gamma=0$ so we use a *Gedankenexperiment*
- Whether or not gravity acts, our object would have had the same angular momentum at $r=\infty$ and hence we can imagine its trajectory without gravity.
- It would fly on a straight line and angular momentum will still be conserved
- It would have a distance of closest approach of h and at this point it would have angular momentum of $|\underline{J}| = hmv_0$

Note: We have gone back to assuming that the central body does not move and is hence assumed infinitely heavy

20.2 Distance of closest approach in hyperbolic orbits (cont.)



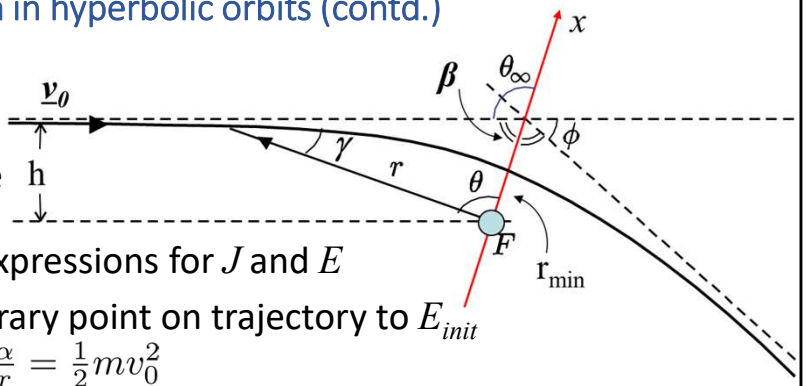
- Another way to get to angular momentum at $r=\infty$:
- Compare: $|\underline{\boldsymbol{J}}| = |\underline{\boldsymbol{r}} \times \underline{\boldsymbol{p}}| = rp\sin(\pi - \gamma) = pr\sin\gamma = ph$
- To the red triangle

$$|\underline{\boldsymbol{J}}| = pr\sin\gamma = ph = hmv_0$$

This is as before:
The magnitude of angular Momentum wrt. to some point P is equal to the product of the impact parameter (or the perpendicular distance of P wrt. to the trajectory) with the magnitude of the momentum

20.2 Distance of closest approach in hyperbolic orbits (contd.)

- So $|\underline{\boldsymbol{J}}_{init}| = hmv_0$
- And $E_{init} = \frac{1}{2}mv_0^2$
- Note: we chose to characterise the initial conditions via h and v_0 because these give simple expressions for J and E



- Now equate energy at an arbitrary point on trajectory to E_{init}

$$E = E_{init} \Leftrightarrow \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r} = \frac{1}{2}mv_0^2$$

- At distance of closest approach we have $r = r_{min}\ ; \dot{r} = 0$

$$\frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}} = \frac{1}{2}mv_0^2$$ use J from above to get $$\frac{(hmv_0)^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}} = \frac{1}{2}mv_0^2 \Rightarrow$$

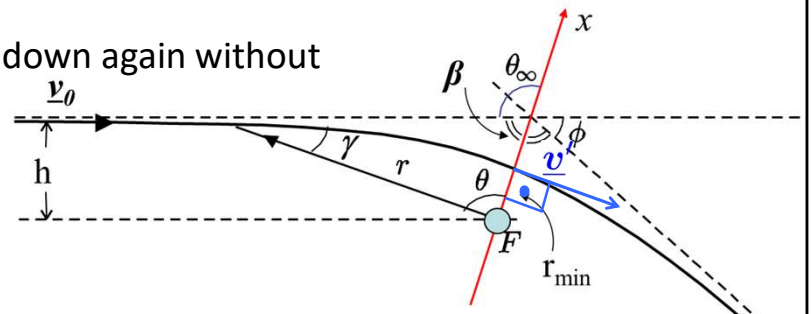
$$\frac{(hmv_0)^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}} = \frac{1}{2}mv_0^2 \overset{\times 2r_{min}^2 \,/mv_0^2}{\Rightarrow} h^2 - \frac{2\alpha r_{min}}{mv_0^2} = r_{min}^2 \Rightarrow$$

$$r_{min} = -\frac{\alpha}{mv_0^2} + \sqrt{(\frac{\alpha}{mv_0^2})^2 + h^2}$$ Does this look familiar?

20.2 Distance of closest approach in hyperbolic orbits (contd.)

- If it is not familiar then write it down again without inserting expressions for J,E
- Rewrite: $\frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}} = E$

  $r_{min}^2 + \frac{\alpha}{E} r_{min} - \frac{J^2}{2mE} = 0$
- This is exactly the same as in L18, p14 calculation of the turning points: $r_{turn}^2 + \frac{\alpha}{E} r_{turn} - \frac{J^2}{2m^2 E} = 0 \Rightarrow r_{min} = -\frac{\alpha}{mv_0^2} + \sqrt{(\frac{\alpha}{mv_0^2})^2 + h^2}$



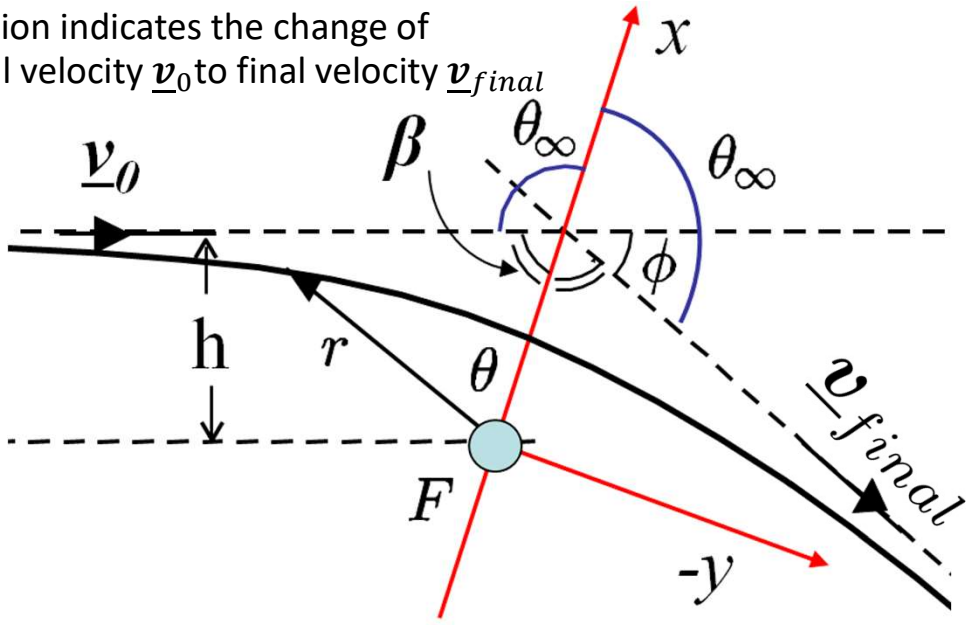
- So we use the expression found there to re-express this in terms of conic-section parameters: $r_{min} = -a(1-e)$ $\quad a = \frac{\alpha}{mv_0^2} \quad e = \sqrt{1 + \left(\frac{hmv_0^2}{\alpha}\right)^2}$

  need minus here to get r_{min}>0 when e>1
- Further find Velocity v' @ r_{min} via angular momentum conservation & $\underline{v}' \perp \underline{r}_{min}$

  $|\underline{J}_{init}| = hmv_0 = mv'r_{min} \Rightarrow v' = \frac{v_o h}{r_{min}}$

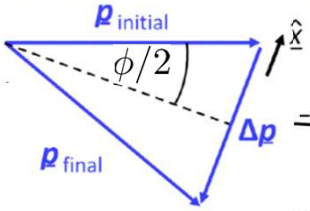
20.3 Angle of deflection in hyperbolic orbits

- The angle of deflection indicates the change of direction from initial velocity $\underline{v}_0$ to final velocity $\underline{v}_{final}$

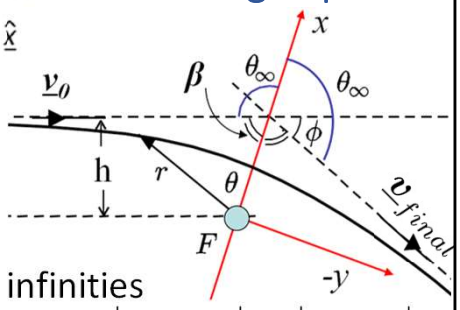


20.3.1 Angle of deflection in hyperbolic orbits: Method 1: using impulse

- Momentum change: $\Delta\underline{p} = \underline{p}_f - \underline{p}_i$



$\Delta p_x = -|\Delta p| \overset{[1]}{=} -2mv_0 \sin\frac{\phi}{2}$



- Note: Since Energy is conserved and we have an infinitely heavy central body which takes up no energy we also conserve the speed of the deflected body at the two infinities
  ➔ conserve the magnitude of initial and final momentum. $|\underline{p}_{initial}| = |\underline{p}_{final}|$
- Use Newton-II: $\frac{dp_x}{dt} = F_x \Rightarrow \Delta p_x = \int_{t=-\infty}^{t=+\infty} F_x dt = \int_{t=-\infty}^{t=+\infty} \frac{-\alpha}{r^2}\cos\theta dt$
- Must change time integral to θ integral since θ is a function of time
- Use: $J = m|\underline{r} \times \underline{v}| = mr\frac{d\theta}{dt}r \Rightarrow dt = \frac{mr^2}{J}d\theta \Rightarrow \Delta p_x = \int_{\theta_{-\infty}}^{\theta_{+\infty}} \frac{-\alpha}{r^2}\frac{mr^2}{J}\cos\theta d\theta$

  projection of v ⊥ r

  symmetric integral of even function

$\Delta p_x = 2\int_0^{\theta_{+\infty}} \frac{-m\alpha}{J}\cos\theta d\theta = \frac{-2m\alpha}{J}\sin\theta_\infty$ Since $\theta_\infty = \frac{\pi}{2} + \frac{\phi}{2} \Rightarrow$

$\Delta p_x \overset{[2]}{=} -\frac{2m\alpha}{J}\cos\frac{\phi}{2}$ divide [2]/[1] ➔ $\cot\frac{\phi}{2} = \frac{Jv_0}{\alpha} = \frac{mhv_0^2}{\alpha} = \frac{hv_0^2}{GM}$

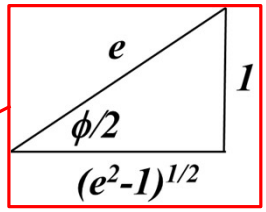
20.3.2 Angle of deflection: Method 2: using hyperbolic orbit parameters

- Orbit equation $r(\theta) = \frac{r_0}{1+e\cos\theta}$

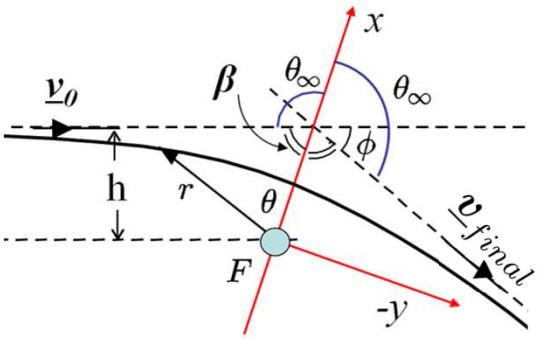
  $r \to \infty \Rightarrow \cos\theta_\infty = -\frac{1}{e}$

From previous slide

$\theta_\infty = \frac{\pi}{2} + \frac{\phi}{2} \Rightarrow \sin\frac{\phi}{2} = \frac{1}{e}$



- Thus: $\cot\frac{\phi}{2} = \sqrt{e^2-1}$ and from p6. $e = \sqrt{1+\frac{2EJ^2}{m\alpha^2}}$
- Which combines to: $\cot\frac{\phi}{2} = \sqrt{\frac{2EJ^2}{m\alpha^2}}$
- We know: $E = \frac{1}{2}mv_0^2$; $J = mv_0h$
- Which gives: $\cot\frac{\phi}{2} = \frac{mv_0^2h}{\alpha} = \frac{v_0^2h}{GM}$
- Which is the same result we found on p.8



20.4 Hyperbolic orbits in Rutherford Scattering

- Consider a light particle of charge $+q$ deviated by the Coulomb repulsion of a heavy nucleus of charge $+Q$. (again assuming a stationary central body!)
- Expect same hyperbolic motion as before but change: $\alpha = GMm \rightarrow \alpha_c = -\frac{Qq}{4\pi\epsilon_0}$
- Now we take the other branch of the hyperbola
- Energy equation now is:

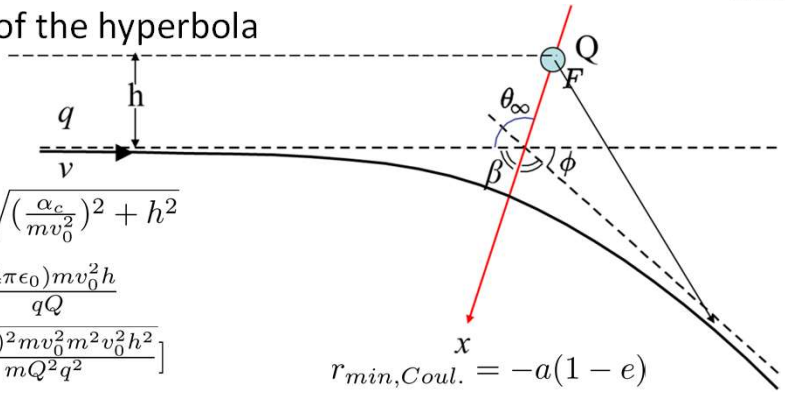
$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha_c}{r}$$

$$\cot\frac{\phi}{2} = \sqrt{\frac{2EJ^2}{m\alpha_c^2}} \quad r_{min} = -\frac{\alpha_c}{mv_0^2} + \sqrt{(\frac{\alpha_c}{mv_0^2})^2 + h^2}$$

$$E = \tfrac{1}{2}mv_0^2 \; ; \; J = mv_0h \quad \cot\frac{\phi}{2} = \frac{(4\pi\epsilon_0)mv_0^2h}{qQ}$$

$$r_{min} = \left(\frac{-qQ}{mv_0^2(4\pi\epsilon_0)}\right)[1 + \sqrt{1 + \frac{(4\pi\epsilon_0)^2mv_0^2m^2v_0^2h^2}{mQ^2q^2}}]$$

$$= \left(\frac{-qQ}{mv_0^2(4\pi\epsilon_0)}\right)[1 + \sqrt{1 + \frac{(4\pi\epsilon_0)^2m^2v_0^4h^2}{Q^2q^2}}]$$



$$r_{min,Coul.} = -a(1-e)$$

$$a = \left(\frac{qQ}{mv_0^2(4\pi\epsilon_0)}\right)$$

$$e = \sqrt{1 + \frac{(4\pi\epsilon_0)^2m^2v_0^4h^2}{Q^2q^2}}$$