The Hilary Term Programme

- Detailed breakdown of all lectures are on the courses canvas page at: https://canvas.ox.ac.uk/files/5129835/download?download_frd=1
- The HT of mechanics is structured into 6 blocks A. to F. as shown below
- The main books I recommend are:
  - Hand and Finch, Analytical Mechanics
  - New & Big from 2021: T.M. Helliwell & V.V. Sahakian, Modern Classical Mechanics (for Lagrangian and Hamiltonian mechanics)
- Other Books I used: Budo, Theoretische Mechanik and Recknagel, Physik. Mechanik (both in German)

A. Newton's law and variable mass and non-inertial reference frames (L. 11-12).

B. Motion of charged particles in B and E fields (L. 13)

C. Vector differentiation, torque, angular momentum and circular motion (L.14-15)

D. Central forces and orbits (L.16-20)

E. Rotational dynamics and rigid bodies (L. 21-24)

F. Lagrangian mechanics (with some Hamiltonian mechanics) (L. 24-29)

Acknowledgement:

Most materials are derived from earlier versions of this course by Neville Harnew and Guy Wilkinson

Classical Mechanics Lecture 11

Newton's 2nd Law & variable mass

Prof Armin Reichold

University of Oxford

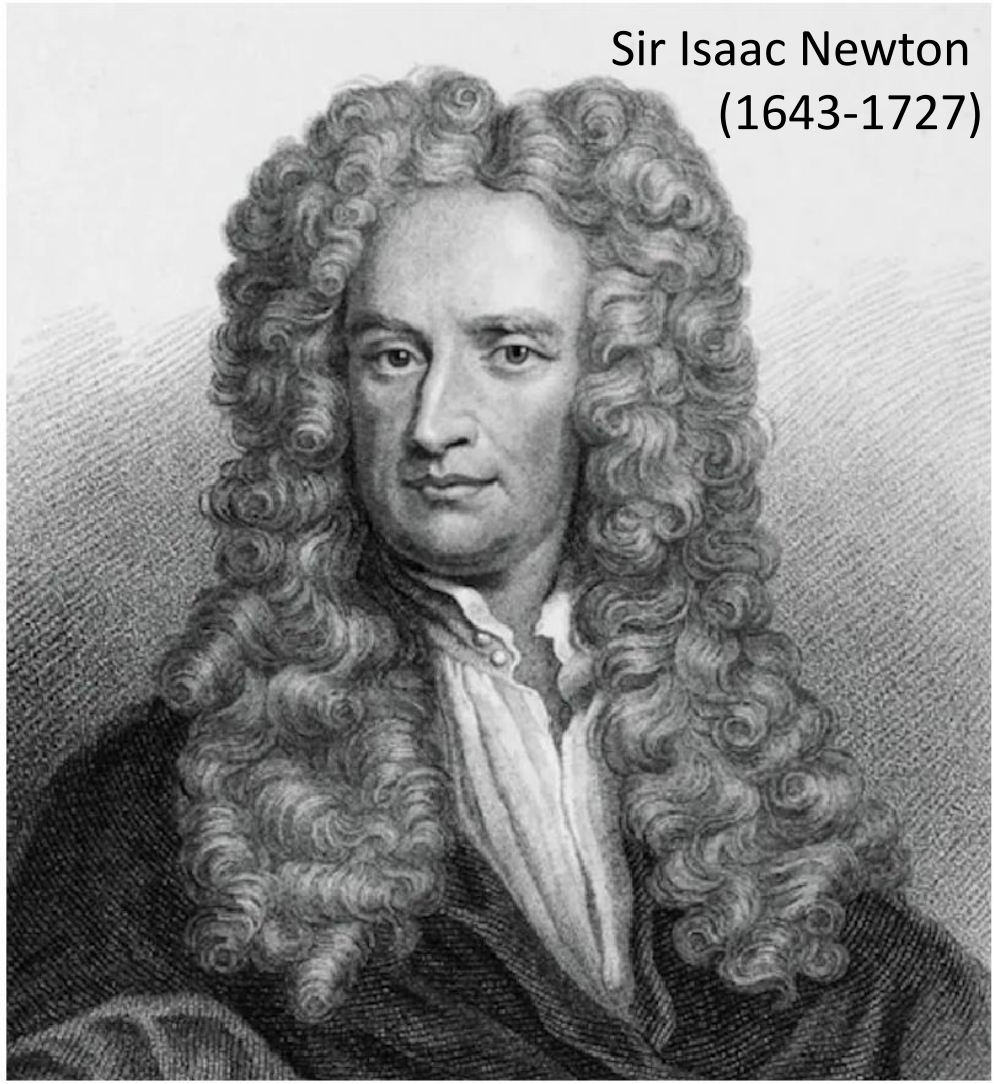
Hilary Term 2023

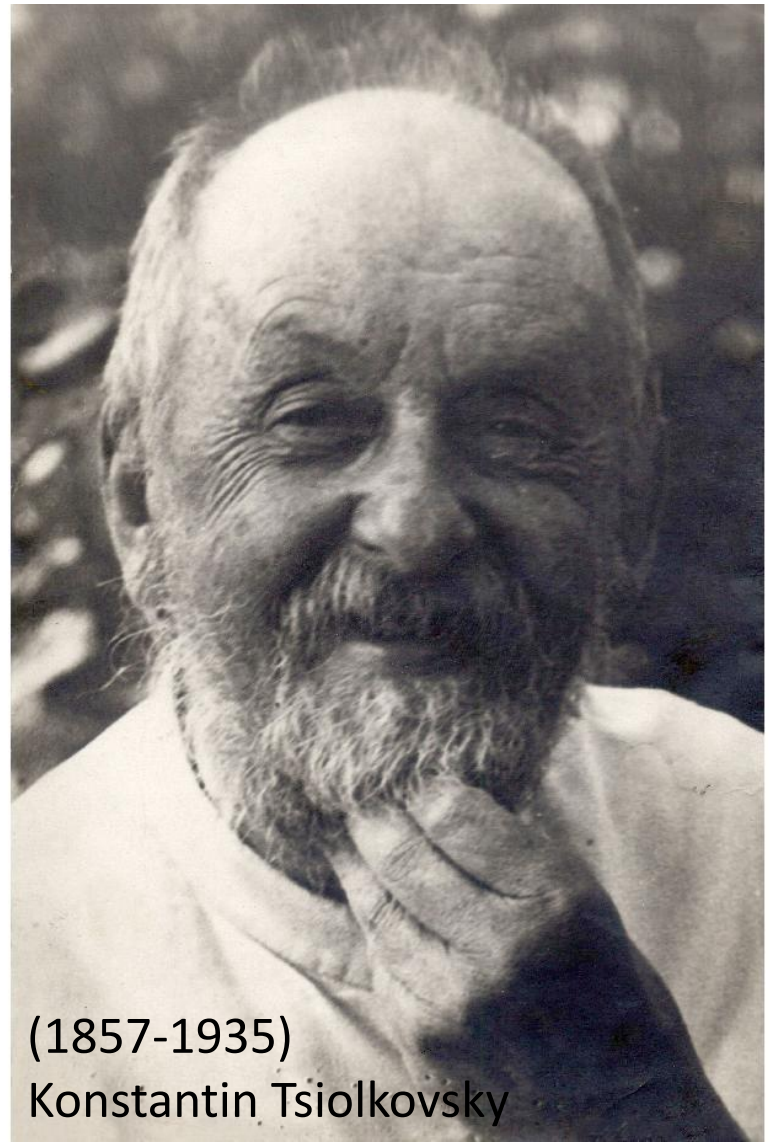
Lecture 11 overview

1. Variable mass: a body acquiring mass continuously
    1. Acquiring mass without external forces
    2. Acquiring mass with external forces
    3. Example acquiring mass: The raindrop
2. Variable mass: a body ejecting mass & the rocket equation
3. 1st Example ejecting mass: The horizontal rocket launch
4. 2nd Example: The vertical rocket launch

Note: To improve clarity I will be using only scalar notations in the section about variable mass.
In other words: no vector notation

The “Heroes” of lectures 11 &12

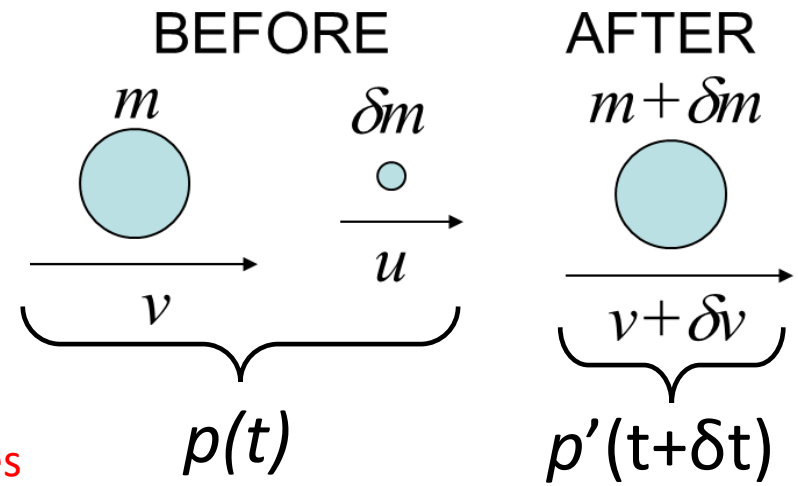

Sir Isaac Newton
(1643-1727)


(1857-1935)
Konstantin Tsiolkovsky

11.1.1 : A body acquiring mass continuously (No external Forces)

- A body of mass m has velocity *v*
- In a short time δt it acquires a small mass *δm*
- *δm* moves along bodies direction with **const.** velocity *u*
- We seek acceleration (EOM) of the body $a = {}^{dv}/_{dt}$
- The change in mass : $m \rightarrow m' = m + \delta m$
- The change in velocity : $v \rightarrow v' = v + \delta v$

!Danger!
Two variables
linked properties



- Case 1: No external force ➔ Newton-II says: change of total momentum $\delta p = p' - p = 0$

$$\delta p = \underbrace{(m+\delta m)(v+\delta v)}_{p'} - \underbrace{(mv+\delta m u)}_{p} = 0$$

doubly small ➔ ignore

relative velocity $w = v - u$

$$\delta p = \cancel{mv} + m\delta v + \delta m v + \overbrace{\delta m \delta v} - \cancel{mv} - u\delta m = 0$$

$$\rightarrow \quad \delta p = m\delta v + \overbrace{(v-u)}\delta m = 0$$

Divide δp by δt which is the time over which the mass is acquired

$$\frac{\delta p}{\delta t} = m\frac{\delta v}{\delta t} + w\frac{\delta m}{\delta t} = 0$$

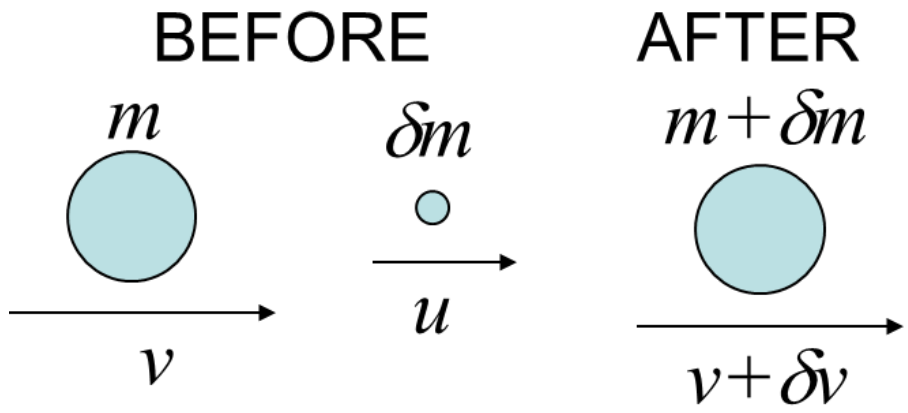
$$\lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = a$$

$$\lim_{\delta t \to 0} \frac{\delta m}{\delta t} = \frac{dm}{dt} = \dot{m}$$

$$a = -\frac{w\dot{m}}{m}$$

Eq 1: Newtons 2nd law with variable mass and without external forces

11.1.2: A body acquiring mass continuously (with external Forces)



relative velocity $w = v - u$

- During time δt a **constant**, total external Force F acts on both bodies.
- As before Newton-II gives change of momentum : $\delta p = F\delta t = \mathrm{m}\delta v + w\delta m$
- As before divide by δt and let $\delta t \to 0$:

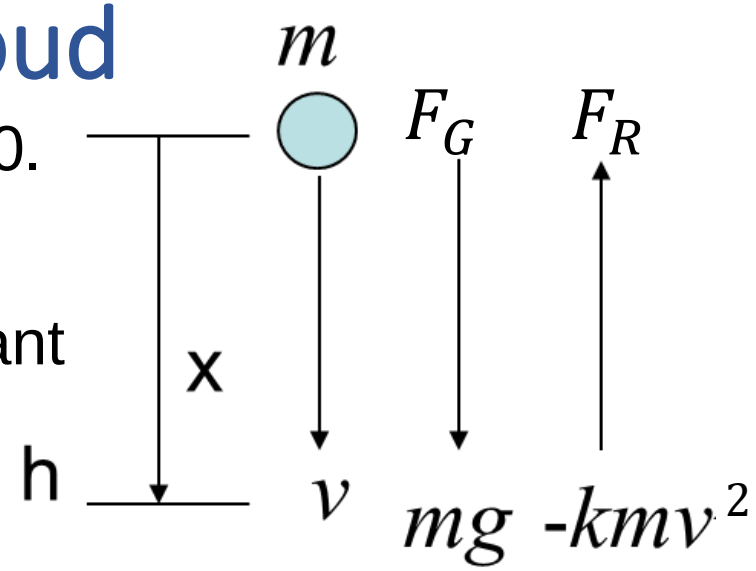
$F = ma + w\dot{m}$ or $a = \frac{F - w\dot{m}}{m}$

Eq 2: Newtons 2nd law with variable mass and external forces

- Note: **ONLY** $w = 0$ or m = const. gives familiar special case $F = ma$
- but Newton-II: $F = \dot{p}$ which we started from is valid in both cases

11.1.3: A Raindrop falling through a cloud

- An idealised raindrop has initial mass=m_0 & initial velocity=0.
- It falls under gravity $F_G = mg$ from height h.
- Air resistance is given by: $F_R = -k\,m\,v^2$ where k is a constant of dimension 1/length.
- Drop acquires mass from vapor or droplets at rest in the air.
- Assume mass accretes as: ${}^{dm}/_{dt} = b\,m\,v$ where b is a constant
- Find EoM & terminal velocity from Newton-2 (Eq. 2): $F_G + F_R = m\frac{dv}{dt} + w\frac{dm}{dt}$



$$mg - kmv^2 = m\frac{dv}{dt} + w\frac{dm}{dt}$$

- vapour is at rest → u = 0 → $w = v$
- ${}^{dm}/_{dt} = b\,m\,v$
- divide by m (makes eqn. indep of m due to simplified models for force and accretion rate)

$$\frac{dv}{dt} = g - kv^2 - bv^2$$ EOM

- terminal velocity when $|{}^{dv}/_{dt}|_{v_t} = 0$

$$v_t = \sqrt{\frac{g}{b+k}}$$

11.1.3: The Raindrop: mass vs. distance fallen

- Let us find the mass of the drop as a function of the distance fallen
- $\frac{dm}{dt} = bmv$ was given
- but we want $\frac{dm}{dx}$ so we can integrate that later. So lets use the chain rule!

$$\frac{dm}{dx} = \frac{dm}{dt}\frac{dt}{dx} \quad \rightarrow \quad \frac{dm}{dx} = bmv\frac{dt}{dx} \quad \rightarrow \quad \frac{dm}{dx} = bmv\,\frac{1}{v} \quad \rightarrow \quad \frac{dm}{dx} = bm$$

- Lets separate variable on this differential equation and integrate it

$$\frac{dm}{m} = bdx \quad \rightarrow \quad \int_{m_0}^{m}\frac{dm}{m} = \int_{0}^{x} bdx \quad \rightarrow \quad [\ln(m)]_{m_0}^{m} = [bx]_0^x \quad \rightarrow \quad \ln\left(\frac{m}{m_0}\right) = bx$$

$$m = m_0 e^{bx}$$

So the mass of the drop grows exponentially with distance. Does it really?

11.1.3: The Raindrop: speed at ground level

- Reuse previous the EoM we found: $\frac{dv}{dt} = g - kv^2 - bv^2$
- as before we have $v(t)$ but we want $v(x)$ so we use chain rule again

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} \xrightarrow{\frac{dx}{dt} = v} \frac{dv}{dt} = \frac{dv}{dx}v \xrightarrow{\frac{dv}{dt} \text{ from EoM}} g - (k+b)v^2 = \frac{dv}{dx}v \xrightarrow{\text{separate variables}}$$

$$\frac{vdv}{g-(k+b)v^2} = dx \xrightarrow{\text{integrate}} \int_0^{v_h} \frac{vdv}{g-(k+b)v^2} = \int_0^h dx \xrightarrow{\text{homework}}$$

$$\left[-\left(\frac{\ln\left(g-(b+k)v^2\right)}{2(b+k)} \right) \right]_0^{v_h} = h \xrightarrow{\text{insert limits \& solve for } v_h}$$

$$v_h = \sqrt{\frac{g}{b+k}\left[1 - \exp\left(-2h(b+k)\right)\right]}$$

- Find terminal velocity from v_h

$$\lim_{h\to\infty} v_h = \sqrt{\frac{g}{b+k}}$$

- as found earlier

11.2: Ejecting Mass and Tsiolkovsky's rocket equation

- A body of mass m has velocity v.
- In short time δt it ejects small mass δm,
- with relative backwards velocity w to the body.
- The systems initial momentum = p
- and final momentum = p'
- Change of systems momentum $\delta p = p' - p$

$$\delta p = \underbrace{\delta m(v-w) + (m-\delta m)(v+\delta v)}_{p'} - \underbrace{mv}_{p}$$

$p' = p_{ejected} + p_{body}$

- multiply out

doubly small, ignore

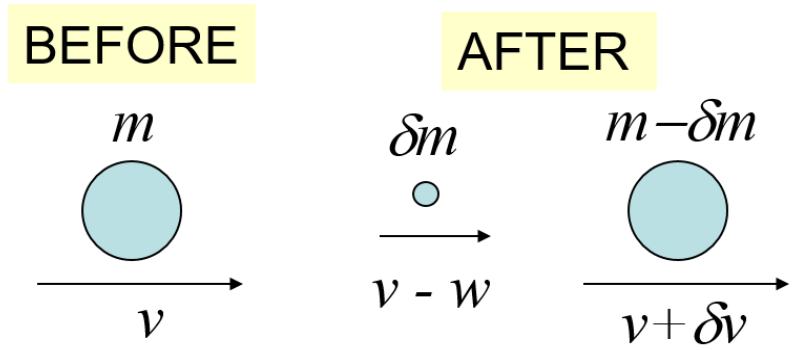
$$\delta p = \delta m v - \delta m w + m v + m\delta v - \delta m v - \overbrace{\delta m \delta v} - m v = -\delta m w + m\delta v$$

- divide by δt & use: $\lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$ & $\lim_{t \to 0} \frac{\delta m}{\delta t} = -\frac{dm}{dt}$
- & remember Newton-II : $\frac{dp}{dt} = F$

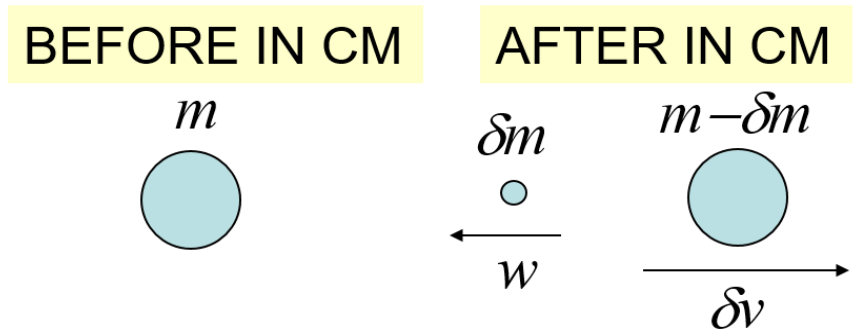
→ Tsiolkovsky rocket Equation aka Newton-2 for variable m

$$F = w\dot{m} + ma$$

Lab Frame



CM Frame looks more like a rocket



11.3: The rocket: launched in empty space $[v(m)]$

- The rocket equation with $F = 0, w = const$:

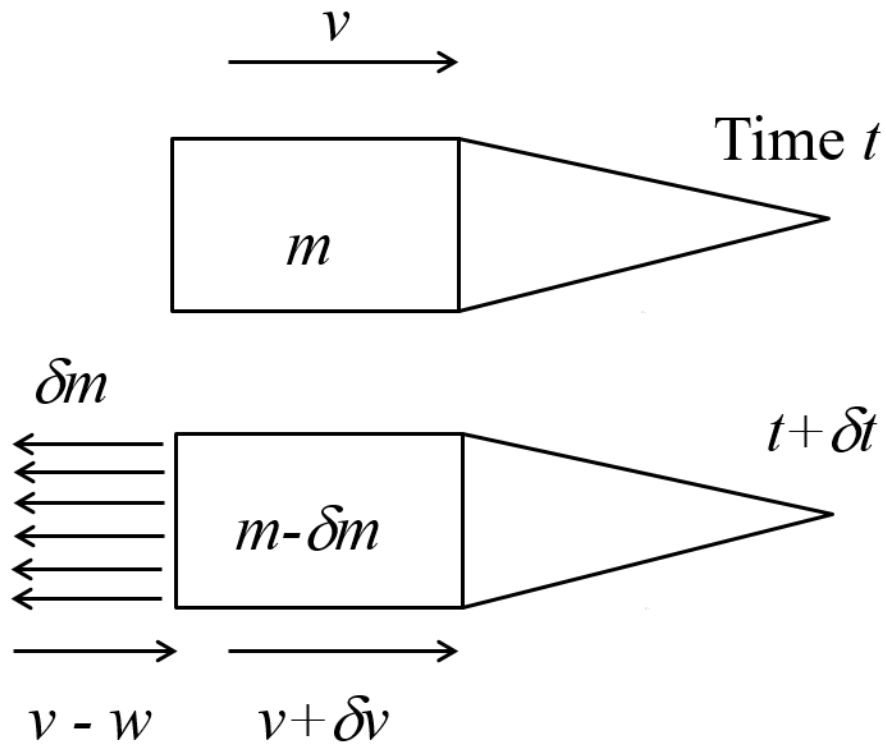
$$0 = m\frac{dv}{dt} + w\frac{dm}{dt}$$

- To find $v(m)$ multiply with dt & separate v and m

$$dv = -w\frac{dm}{m}$$

- Initial and final velocities are v_i and v_f
- Initial and final masses are m_0 and m_f
- Integral of the above:

$$\int_{v_i}^{v_f} dv = \int_{m_0}^{m_f} -w\frac{dm}{m}$$



- Integrate and put in limits:

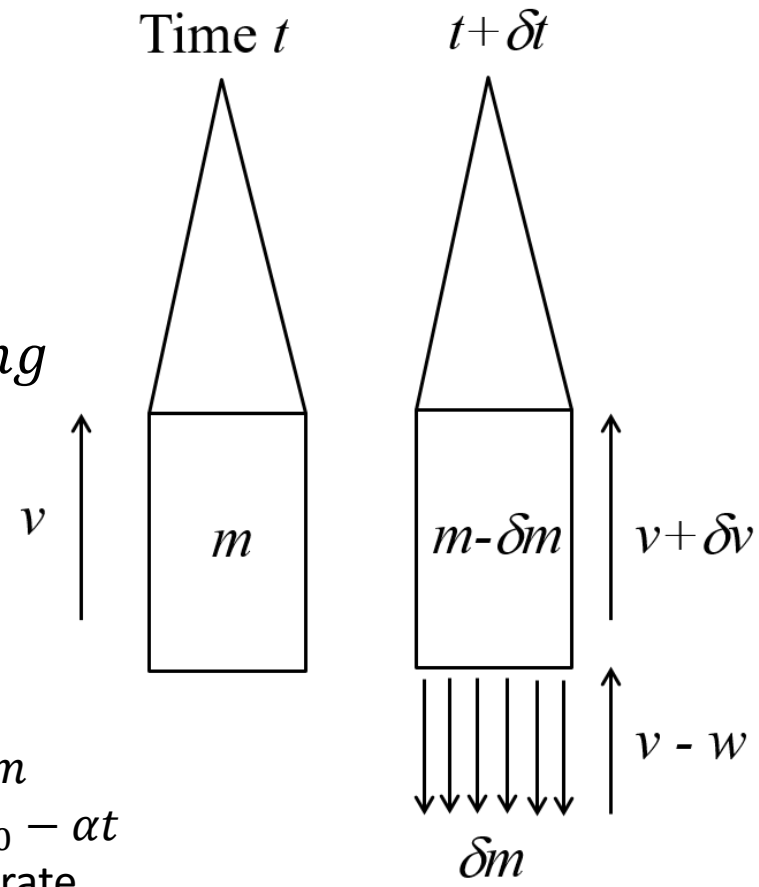
$$\Delta v = v_f - v_i = w \ln\left(\frac{m_0}{m_f}\right)$$

THE BANE OF ROCKETRY

- Shows that gain in speed depends on log of rockets initial to final mass ratio

11.4 The rocket: vertical launch $[v(t)]$

- Remember the rocket equation: $F = m\frac{dv}{dt} + w\frac{dm}{dt}$
- External force F is gravity (neglecting drag, i.e. launch from moon)
- Small change of distance to earth during burn ➔ $g = const$ ➔ $F = -mg$
- Mass ejected at constant relative speed: $w = const$
- Mass ejected at constant rate: $^{dm}/_{dt} = -\alpha = const$ ➔ $m = m_0 - \alpha t$
  where m_0 = initial total rocket mass
- Thrust: Rate at which rocket "ejects momentum" $F_T = w\alpha$
- Rocket launched vertically upward: Find $v(t)$



$\div m$

$m = m_0 - \alpha t$

integrate

$$-mg = m\frac{dv}{dt} - w\alpha \rightarrow \quad mdv = (-mg + w\alpha)dt \rightarrow$$

separate variables

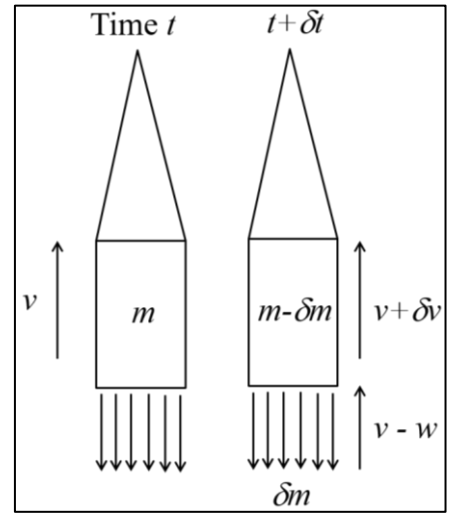
Note: m=m(t)

$$\int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} \left(-g + \frac{w\alpha}{m_0 - \alpha t}\right) dt \rightarrow v_f - v_i = \left[\underbrace{-g(t_f - t_i)}_{\Delta t} - \underbrace{w \ln \frac{m_0 - \alpha t_f}{m_0 - \alpha t_i}}_{\frac{m_f}{m_0}}\right]$$

speed after free falling for Δt

speed a free rocket achieves in Δt

11.4 The rocket: vertical launch continued



- Seeking x(t) from $v_f - v_i = \left[-g(t_f - t_i) - w \ln \frac{m_0 - \alpha t_f}{m_0 - \alpha t_i}\right]$
- simplify notation and set $t_i = 0$ and $v_i = 0$ and $v_f = v$ and $v = \frac{dx}{dt}$

$$\frac{dx}{dt} \underset{[1]}{=} -gt - w \ln \frac{m_f}{m_0} = -gt - w \ln\left(1 - \frac{\alpha}{m_0} t\right)$$

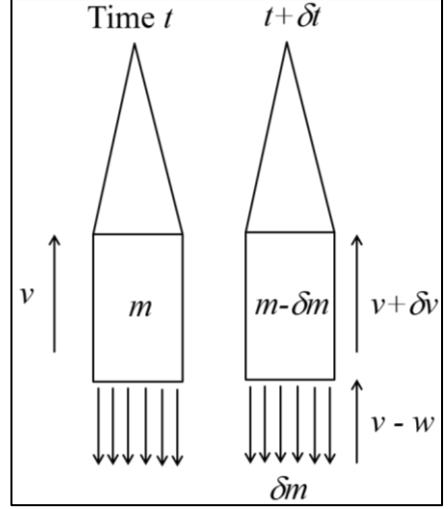
- Initially find minimum w at given α for rocket to rise immediately, from rocket eqn.

$$F = m\frac{dv}{dt} + w\frac{dm}{dt}$$

$$-mg = m\frac{dv}{dt} - w\alpha \rightarrow \left.\frac{dv}{dt}\right|_{t=0} = -g + \frac{w\alpha}{m_0} \overset{!}{>} 0 \rightarrow w_{min} > \frac{m_0 g}{\alpha} \quad (w\alpha)_{min} > m_0 g$$

- Define some useful terms:
- Fuel fraction: $f = \frac{m_{fuel}}{m_0}$ → final mass: $m_f \underset{[2]}{=} (1 - f)m_0$
- Burn-out time: $m_f \underset{[3]}{=} m_0 - \alpha t_f$ → $T = t_f = \frac{m_0 f}{\alpha}$

11.4 The rocket: vertical launch continued



- recall: $\frac{dx}{dt} \overset{[1]}{=} -gt - w\ln\frac{m_f}{m_0} = -gt - w\ln\left(1-\frac{\alpha}{m_0}t\right)$
- and $T \overset{[2]}{=} \frac{m_0 f}{\alpha}$ and $m_f \overset{[3]}{=} (1-f)m_0$
- Find $v(T) = v_{max}$ and $x(T)$ at burn-out :
  - v_{max} from [1] [2] [3] → $v_{max} = -\frac{gm_0 f}{\alpha} - w\ln(1-f)$
  - $x(T)$ from [1] & sep. vars. & integrate → $\int_0^{x_f} dx = \int_0^T -gt - w\ln\left(1-\frac{\alpha}{m_0}t\right)dt$
  - Use standard integral: $\int \ln z\,dz = z\ln z - z$ with $z = \left(1-\frac{\alpha}{m_0}t\right) \rightarrow \frac{dz}{dt} = -\frac{\alpha}{m_0} \rightarrow dt = -\frac{m_0}{\alpha}dz$

$x_f = \frac{-gT^2}{2} + \frac{wm_0}{\alpha}\left[\left(1-\frac{\alpha}{m_0}t\right)\ln\left(1-\frac{\alpha}{m_0}t\right) - \left(1-\frac{\alpha}{m_0}t\right)\right]_0^T$ eliminate T via [2] & simplify →

$$x_f = \frac{wm_0}{\alpha}\underbrace{\left[\underbrace{(1-f)}_{>0}\underbrace{\ln(1-f)}_{<0} + \underbrace{f}_{>0}\right]}_{<0} - \frac{gm_0^2 f^2}{2\alpha^2}$$

distance the rocket would have free fallen in time T is < 0

Ideal rocket:

$$\lim_{f\to 1} x_f = \frac{wm_0}{\alpha} - \frac{gm_0^2}{2\alpha^2}$$

Useless rocket:

$$\lim_{f\to 0} x_f = 0 - 0$$

