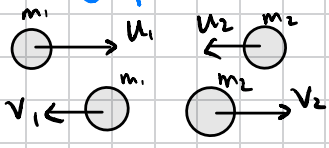
CP1. PS2. Classical Mechanics.

Introductory problems. Oct. 17. 2025

1. (a)



ZMF: $\begin{cases} m_1u_1 + m_2u_2 = 0 \ ① \\ m_1v_1 + m_2v_2 = 0 \ ② \\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \end{cases}$

$\Rightarrow m_1u_1^2 - m_1v_1^2 = m_2v_2^2 - m_2u_2^2$

$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$

① − ②: $m_1(u_1 - v_1) = m_2(v_2 - u_2)$

Thus: $u_1 + v_1 = v_2 + u_2$

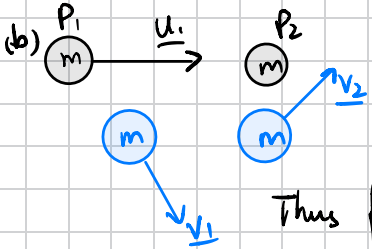
$u_1 - u_2 = v_2 - v_1$

From ① & ②: $\begin{cases} u_2 = -\frac{m_1}{m_2}v_1 \\ v_2 = -\frac{m_1}{m_2}u_1 \end{cases}$

$u_1\left(1 + \frac{m_1}{m_2}\right) = -\left(1 + \frac{m_1}{m_2}\right)v_1$

$\Rightarrow u_1 = -v_1 \qquad u_2 = -v_2$ The magnitude doesn't change. □

(b)



LAB:

$\begin{cases} \sum p = \text{cons.} \Rightarrow m\underline{u}_1 = m\underline{v}_1 + m\underline{v}_2 \\ \sum E = \text{cons.} \Rightarrow \frac{1}{2}m|\underline{u}_1|^2 = \frac{1}{2}m|\underline{v}_1|^2 + \frac{1}{2}m|\underline{v}_2|^2 \end{cases}$

Thus $\begin{cases} \underline{u}_1 = \underline{v}_1 + \underline{v}_2 \ ① \\ |\underline{u}_1|^2 = |\underline{v}_1|^2 + |\underline{v}_2|^2 \ ② \end{cases}$

Take ① into ②: $|\underline{v}_1 + \underline{v}_2|^2 = |\underline{v}_1|^2 + |\underline{v}_2|^2$

$|\underline{v}_1|^2 + 2|\underline{v}_1 \cdot \underline{v}_2| + |\underline{v}_2|^2 = |\underline{v}_1|^2 + |\underline{v}_2|^2 \Rightarrow \boxed{\underline{v}_1 \cdot \underline{v}_2 = 0}$

If P_2 comes with same direction as P_1 initially goes, then P_1 comes to rest. Otherwise P_1 has to move to keep momentum conserved on its initial velocity direction.

From $\begin{cases} \underline{u}_1 = \underline{v}_1 + \underline{v}_2 \\ |\underline{u}_1|^2 = |\underline{v}_1|^2 + |\underline{v}_2|^2 \end{cases} \Rightarrow \begin{cases} \underline{u}_1 - \underline{v}_1 = \underline{v}_2 \ ③ \\ |\underline{u}_1|^2 - |\underline{v}_1|^2 = |\underline{v}_2|^2 \end{cases}$

③ $\Rightarrow |\underline{u}_1 - \underline{v}_1|^2 = |\underline{u}_1|^2 + |\underline{v}_1|^2 - 2|\underline{u}_1 \cdot \underline{v}_1| = |\underline{u}_1|^2 - |\underline{v}_1|^2$

$2|\underline{v}_1|^2 = 2|\underline{u}_1 \cdot \underline{v}_1|$

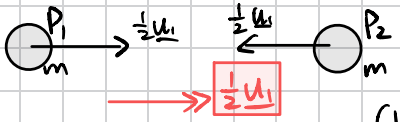
$|\underline{v}_1| = |\underline{u}_1 \cdot \underline{v}_1|$

Since $|\underline{v}_1| > 0 \Rightarrow |\underline{u}_1 \cdot \underline{v}_1| > 0$

$\underline{u}_1 \cdot \underline{v}_1 = |\underline{u}_1||\underline{v}_1|\cos\theta > 0$

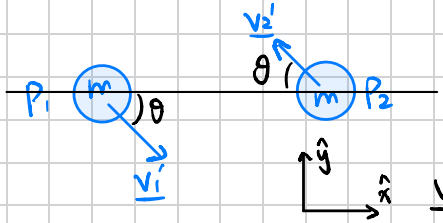
$\Rightarrow \theta < 90°$

(c) ZMF



$\sum p = m u_1 \Rightarrow \boxed{V_{cm} = \frac{1}{2} u_1}$

$$\begin{cases} |\underline{V_2'}| = |\underline{V_1'}| = \left|\frac{1}{2}\underline{u_1}\right| \\ \underline{V_2'} = -\underline{V_1'} \end{cases}$$



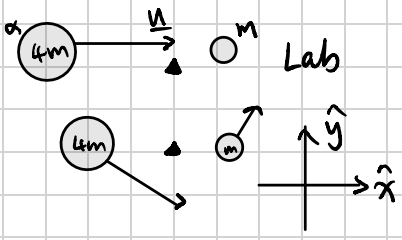
$$\Rightarrow \underline{V_1'} = \frac{u_1}{2}\begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix} \quad \underline{V_2'} = \frac{u_1}{2}\begin{pmatrix}-\cos\theta \\ \sin\theta\end{pmatrix}$$

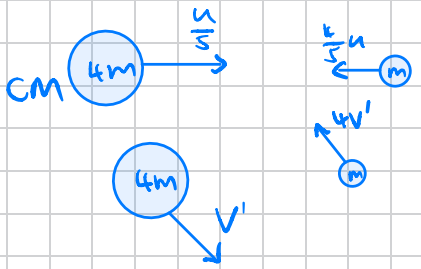
$$\underline{V_1}\cdot\underline{V_2} = (\underline{V_1'} + \tfrac{1}{2}\underline{u_1})\cdot(\underline{V_2'} + \tfrac{1}{2}\underline{u_1})$$

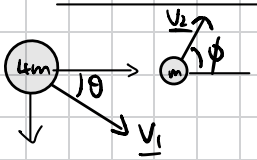
$$= \underline{V_1'}\cdot\underline{V_2'} + \tfrac{1}{2}\underline{u_1}\cdot(\underline{V_1'} + \underline{V_2'}) + \tfrac{1}{4}\underline{u_1}\cdot\underline{u_1}$$

$$= \tfrac{1}{4}|\underline{u_1}|^2 - \tfrac{1}{4}|\underline{u_1}|^2 = 0$$

2.







$$\begin{cases} 4m v_1\cos\theta + m v_2\cos\phi = 4mu \\ 4m v_1\sin\theta + m v_2\sin\phi = 0 \\ \frac{1}{2}4m v_1^2 + \frac{1}{2}m v_2^2 = \frac{1}{2}m\,4u^2 \end{cases} \Rightarrow \begin{cases} 4v_1\cos\theta + v_2\cos\phi = 4u \\ 4v_1\sin\theta + v_2\sin\phi = 0 \\ 4v_1^2 + v_2^2 = 4u^2 \end{cases}$$

$$v_2\cos\phi = 4u - 4v_1\cos\theta \qquad v_2\sin\phi = -4v_1\sin\theta$$

$$v_2^2 = 16(u - v_1\cos\theta)^2 + 16 v_1^2\sin^2\theta$$

$$\rightarrow 4v_1^2 + 16u^2 - 32uv_1\cos\theta + 16v_1^2 = 4u^2$$

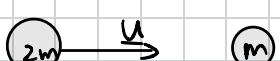
$$12u^2 - 32uv_1\cos\theta + 20v_1^2 = 0$$

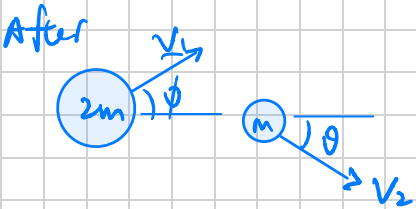
$$\boxed{3u^2 - 8uv_1\cos\theta + 5v_1^2 = 0}$$

$$\Delta = b^2 - 4ac = (8\cos\theta)^2 - 3\cdot4\cdot5 = 0 \Rightarrow \cos^2\theta = \frac{60}{64} = \frac{15}{16}$$

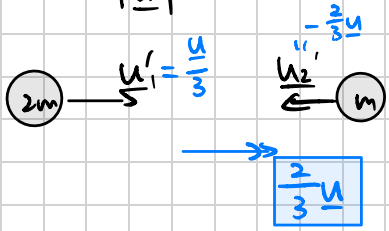
$$\cos\theta_{extre} = \frac{\sqrt{15}}{4} \Rightarrow \boxed{\theta \simeq 14.5^\circ}$$

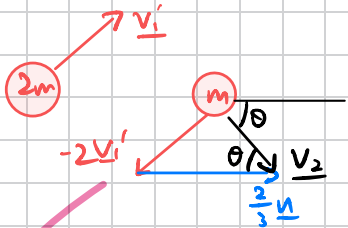
3. LAB





$$\alpha = \frac{|\underline{v}_2 - \underline{v}_1|}{|\underline{u}|} \Rightarrow |\underline{v}_2 - \underline{v}_1| = \alpha |\underline{u}| \quad \text{In CM frame.}$$





$$\alpha = \frac{|\underline{v}_1' + 2\underline{v}_1'|}{|\underline{u}|} = 3\frac{|\underline{v}_1'|}{|\underline{u}|} \Rightarrow 3|\underline{v}_1'| = \alpha|\underline{u}| \Rightarrow \boxed{|\underline{v}_2'| = \frac{2}{3}\alpha|\underline{u}|}$$



$$2v_1\sin\phi = v_2\sin\theta$$
$$2v_1\cos\phi + v_2\cos\theta = 2u$$
$$\Rightarrow \underline{v_2\cos\theta = 2(u - v_1\cos\phi)}$$

I thought this may useful but not.

Cosine Rule:

$$0 = v_2^2 + (1-\alpha^2)\left(\frac{2}{3}u\right)^2 - 2(v_2)\left(\frac{2}{3}u\right)\cos\theta$$

$$\frac{9}{4}\underline{v_2^2} - 3\underline{u}\cos\theta\,\underline{v_2} + (1-\alpha^2)\underline{u}^2 = 0$$

$$v_2 = \frac{2}{3}u\left(\cos\theta \pm \sqrt{\cos^2\theta - (1-\alpha^2)}\right)$$

$$= \boxed{\frac{2}{3}u\left(\cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}\right)}$$

4. a. $F(x) = m\frac{dv}{dt}$

$$\int_{x=a}^{x=b} F(x)\,dx = \int_{x=a}^{x=b} m\frac{dv}{dt}\,dx$$

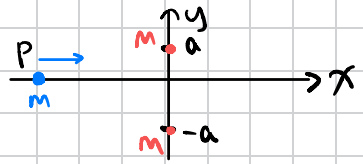
$$W_{ab} = \int_{v=v_a}^{v=v_b} m\,v\,dv = \frac{1}{2}mv_a^2 - \frac{1}{2}mv_b^2 = T(b) - T(a)$$

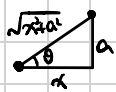
It is universal, not only to conservative force.

b. $F(x) = \frac{k}{x^2}$ $\quad V(x=x_0) = u$ $\quad T = \frac{1}{2}mu^2$.

Assume minimum approach is to $x = x_1$ $\quad \boxed{x_1 = \frac{2kx_0}{2k - x_0mu^2}}$

$$\int_{x_0}^{x_1} F(x)\,dx = \left[-kx^{-1}\right]_{x_0}^{x_1} = -k/x_1 + k/x_0 = \frac{1}{2}mu^2$$

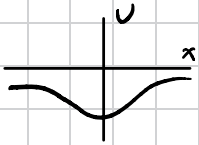
5. 

(a) 

$F = 2 \cdot F_x = -2 \cdot \frac{GMm}{r^2}\cos\theta$

$= -2\frac{GMm}{x^2+a^2}\frac{x}{\sqrt{x^2+a^2}} = \boxed{\frac{-2GMmx}{(x^2+a^2)^{\frac{3}{2}}}}$

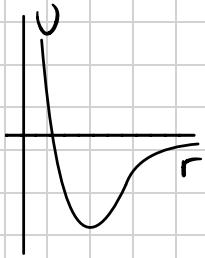
b. $U(x) = \int_0^x -F(x')\,dx' = \int_0^x \frac{2GMmx'}{(x'^2+a^2)^{\frac{3}{2}}}\,dx' = \boxed{-\frac{2mMG}{\sqrt{a^2+x^2}}}$



c. $T(x) = E - U(x)$ and $E = U(\frac{3a}{4}) = -\frac{8mMG}{5a}$

$T(x) = mMG\left(\frac{2}{\sqrt{a^2+x^2}} - \frac{8}{5a}\right)$ $\quad T'(x) = -\frac{2x}{(x^2+a^2)^{\frac{3}{2}}}$ Take $x=0$

$T(0) = \frac{2mMG}{5a} = \frac{1}{2}mv^2 \Rightarrow \boxed{v = \sqrt{\frac{4MG}{5a}}}$

6. (a) 

$U(r) = \varepsilon\left(r_0^{12}r^{-12} - 2r_0^6 r^{-6}\right)$

$U'(r) = \varepsilon\left(-12r_0^{12}r^{-6} + 12r_0^6\right)r^{-7} = 0$

$\Rightarrow \boxed{r = r_0,\ U(r_0) = \varepsilon}$

(b) $U' = 12r_0^6\varepsilon\left(r^{-7} - r_0^6 r^{-13}\right) \Rightarrow U'(r_0) = 12r_0^6\varepsilon(r_0^{-7} - r_0^{-7}) = 0$

$U'' = 12r_0^6\varepsilon\left(-7r^{-8} + 13r_0^6 r^{-14}\right) \Rightarrow U''(r_0) = 72r_0^{-2}\varepsilon$

$f(x)\big|_a = f(a) + \frac{f'(a)}{1!}x + \frac{f''(a)}{2!}x^2 + \cdots \Rightarrow U(r \simeq r_0) \simeq \varepsilon + \frac{36}{r_0^2}\varepsilon(r-r_0)^2 + \cdots$

(c) Take $U(r_0) = 0$ thus define $x = r - r_0$

$U(x) = \left(\frac{36}{r_0^2}\varepsilon\right)x^2 \quad -U'(x) = F(x) = -\left(\frac{72}{r_0^2}\varepsilon\right)x = m\ddot{x}$

This m here should μ instead.

It represents REDUCED MASS. since this is a two-body motion.

$-\omega^2 x = \mu\ddot{x}$ where $\mu = \frac{m}{2}$. $\Rightarrow \omega = \sqrt{\frac{144\varepsilon}{r_0^2 m}} = \frac{12}{r_0}\sqrt{\frac{\varepsilon}{m}}$

(d) $\omega = 2\pi f = 2\pi \times 3\times10^{13} = 2\times10^{14}\,s^{-1}$ $\quad \omega = \sqrt{\frac{k}{m}}$ $\quad m = 14\times1.66\times10^{-27}$

$= 20\times10^{-27}\,kg$

$k = \mu\omega^2 = 10\times10^{-27}\times4\times10^{28} \sim 400\,Nm^{-1}$

7. No force on other directions but only vertically downwards. Thus restricted to the single plane.

(a) $T = \frac{2V}{g}\sin\theta$ (c) $R = \frac{2V^2\sin\theta\cos\theta}{g}$

(b) $H = \frac{V^2\cos^2\theta}{2g}$ (d) $T = \frac{1}{2}mV^2\sin^2\theta$ $U = \frac{m}{2}V^2\cos^2\theta$

$\theta = 45°$ $\sin\theta = \frac{\sqrt{2}}{2}$ $\cos\theta = \frac{\sqrt{2}}{2}$ $E = \frac{1}{2}mV^2$ consistent.

V	R (real/expect)	T
101.8	972/1056.4	14.4/14.7
112.2	1160/1283.2	15.7/16.2
121.9	1349/1514.7	17.0/17.6
131.4	1539/1760.0	18.2/18.9

Due to air resistance, Both R and T are lower than expected value. (Since always opposite to V)

8. $\Delta E = U - T = 1\times9.81\times100 - 0.5\times1\times40^2 = 181\,J \Rightarrow F = 1.81\,N$

'Average' means dividing the overall distance travelled.

9. $d = 0.3m$ $m = 0.5\times10^{-3}kg$

(a) $W = F \Rightarrow mg = 6\pi a\eta v$ $v = \frac{mg}{6\pi a\eta} = \frac{0.5\times10^{-3}\times9.81}{6\pi\times0.15\times1.7\times10^{-5}} = \boxed{102\,ms^{-1}}$

(b) $F = W - f = m\dot{v} \Rightarrow mg - kv = m\dot{v}$

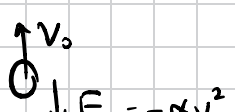
$\boxed{\dot{v} = g - \lambda v}$ $\boxed{\lambda = \frac{6\pi a\eta}{m}}$

$\int\frac{dv}{g-\lambda v} = \int dt$ Take $t=0, v=0$

$-\frac{1}{\lambda}\ln(g-\lambda v) = t + C \Rightarrow v(t) = \frac{g}{\lambda}(1 - e^{-\lambda t})$ $e^{-\lambda t} = 5\% \Rightarrow \boxed{t = 31.2s}$

(c) It can reach terminal speed earlier. (much more earlier). I will choose original statement.

10. (a) $F = m\ddot{x} = -mg - \alpha v^2$



$\beta = \frac{\alpha}{m}$ $\boxed{\frac{dv}{dt} = -g - \beta v^2}$ $\Rightarrow$ $\boxed{v\frac{dv}{dx} = -g - \beta v^2}$

$$\int_0^{v_0} \frac{v\,dv}{g+\beta v^2} = \int dx$$

$$\frac{1}{2\beta}\ln\left(\frac{1}{g+\beta v^2}\right) = x + C$$

Take $v_t = \sqrt{\frac{mg}{\alpha}}$

$a = \frac{v_t^2}{2g}$

$$C = \frac{1}{2\beta}\ln\left(\frac{1}{g+\beta v_0^2}\right)$$

$$h = \frac{1}{2\beta}\ln\left(\frac{1}{g}\right) - \frac{1}{2\beta}\ln\left(\frac{1}{g+\beta v_0^2}\right)$$

$$= \frac{1}{2\beta}\ln\left(\frac{g+\beta v_0^2}{g}\right) = \boxed{\frac{m}{2\alpha}\ln\left(1+\frac{\beta}{g}v_0^2\right)} = a\ln\left(1+\left(\frac{v_0}{v_t}\right)^2\right)$$

(b) $$v\frac{dv}{dx} = g - \beta v^2$$

$$\int_0^{v_r} \frac{v\,dv}{g-\beta v^2} = \int_0^h dx$$

$$\left[-\frac{1}{2\beta}\ln(g-\beta v^2)\right]_0^{v_r} = h$$

$$-a\ln\left(1-\frac{\beta}{g}v_r^2\right) = h$$

$$1-\frac{\beta}{g}v_r^2 = e^{-h/a}$$

$$1-e^{-h/a} = \frac{\beta}{g}v_r^2$$

$$\frac{g}{\beta}\left(1-e^{-h/a}\right) = v_r^2 = \boxed{v_t^2\left(1-e^{-h/a}\right)}$$