Functions of a complex variable (S1)

Problem sheet 1

I. Complex numbers; elementary functions; sets in the complex plane

1. Determine all fifth roots of unity: $w = \sqrt[5]{1}$.
2. Determine all cubic roots of $z = i - 1$: $w = (i-1)^{1/3}$.
3. Solve the equation $(z+i)^5 + (z-i)^5 = 0$.
4. Find the real and imaginary parts of (a) $\cos i$ and (b) $\sin i$.
5. Verify that for any two complex numbers z_1 and z_2

$$|z_1 + z_2| \le |z_1| + |z_2| \quad , \qquad |z_1 + z_2| \ge |\,|z_1| - |z_2|\,| \qquad (\text{"triangle inequalities"}) \quad .$$

6. Solve the equations

$$\text{(a)}\ \ e^z = -1 \quad , \qquad \text{(b)}\ \ e^z = -2 \quad , \qquad \text{(c)}\ \ \sinh z = 0 \quad , \qquad \text{(d)}\ \ \sin z = 3 \quad .$$

7. Draw the curves in the complex z plane

$$\text{(a)}\ \ |z-1| = 1 \quad , \qquad \text{(b)}\ \ (\text{Re}\ z)^2 + 2\,(\text{Im}\ z)^2 = 1 \quad , \qquad \text{(c)}\ \ \text{Im}\ z^2 = 2 \quad .$$

8. Find all the values of

$$\text{(a)}\ \ i^i \quad , \qquad \text{(b)}\ \ (1+i)^i \quad , \qquad \text{(c)}\ \ (-1)^{1/\pi} \quad .$$

9. Find the real and imaginary parts of the principal branch of the logarithm of

$$\text{(a)}\ \ z = 1 - i \quad , \qquad \text{(b)}\ \ z = 1 + i \quad , \qquad \text{(c)}\ \ z = i \quad .$$

10. Consider the set $S = \left\{ \frac{i}{n} : n = 1, 2, \ldots \right\}$ in the complex plane. (a) What are the limit points of S? (b) What are the interior and boundary points of S? (c) State whether or not S is open; closed; bounded; connected; compact.

II. Complex differentiation; holomorphic functions

11. Show that the function $f(z) = z^2$ is holomorphic in the entire complex plane, while the function $f(z) = |z|^2$ is holomorphic nowhere.
12. Determine the subsets of the complex z plane in which each of the following functions is holomorphic,

$$\text{(a)}\ \ \frac{1}{e^z - 1} \quad , \qquad \text{(b)}\ \ \frac{1}{(1+z^2)^2} \quad , \qquad \text{(c)}\ \ \cos \overline{z} \quad , \qquad \text{(d)}\ \ \tan z \quad ,$$

and calculate the derivative of each function in its region of holomorphy.

13. Let the function f be holomorphic on the domain D in $\mathbb{C}$.

    (a) Show that if f is real-valued then f must be constant.

    (b) Show that if $|f|$ is constant then f must be constant.

14. (a) Determine which of the following functions $u(x, y)$ are harmonic:

    (i) $x^2 - y^2 - x$, (ii) $\sin x \cosh y$, (iii) $e^{-x} \cos y + xy$, (iv) $\sin x - \cos y$, (v) $x - y$.

    (b) For each of the harmonic functions above, find a holomorphic function of which it is the real part, and find a harmonic conjugate function $v(x, y)$.

15. Take real-valued function u on domain D in $\mathbb{C}$, $u : D \to \mathbb{R}$. Let u be harmonic on D. Show that if u^2 is harmonic on D then u is constant.

16. (a) Find the holomorphic function whose imaginary part is given by

$$e^{-x}(x \cos y + y \sin y)$$

    and which vanishes at the origin.

    (b) Determine the family of curves in the xy plane which are orthogonal to the curves

$$e^{-x}(x \cos y + y \sin y) = \text{const.} \ .$$

17. Locate and classify the singular points of the following functions, and determine whether they are isolated:

$$\text{(a)} \ \frac{\cos z}{(z+i)^3} \ , \qquad \text{(b)} \ \frac{z^2}{(z^2-1)(z^2+4)} \ , \qquad \text{(c)} \ \frac{1}{\sin 1/z} \ .$$

18. Classify the behaviour of the following functions at $z = \infty$:

$$\text{(a)} \ z(1+z^2) \ , \qquad \text{(b)} \ e^z \ , \qquad \text{(c)} \ \frac{1+z^2}{z^2} \ .$$

19. Consider the mapping specified by the function

$$f : \ z \mapsto w = z + \frac{1}{z} \ .$$

    (a) Give the subset of the complex z plane in which f is holomorphic.

    (b) Determine whether the mapping is conformal in the region of holomorphy.

    (c) Onto which subset of the w plane does f map the upper half and lower half of circle $|z| = 1$?

    (d) What is the image through f of a circle $|z| = \rho$ with $\rho \neq 1$?

20. Consider the mapping specified by the function

$$f : \ z \mapsto w = \frac{i - z}{i + z} \ .$$

    (a) Is the mapping conformal?

    (b) What is the image through f of the real axis $z = x$?

    (c) Onto which subset of the w plane does f map the upper half plane Im $z > 0$?

Functions of a complex variable (S1)

Problem sheet 2

I. Multi-valued functions; branch points and branch cuts

1. (a) Find the location and order of the branch points of the function

$$w = (z-1)^{1/3}$$

and describe a branch cut. (b) Describe a Riemann surface for this function, and determine the image of each Riemann sheet in the w plane.

2. For each of the following functions

$$(a)\ \ln\left(\frac{z-1}{z+1}\right) \quad , \qquad (b)\ \frac{\ln(z+i)}{1+z^2} \quad , \qquad (c)\ \ln(z^2-1) \quad ,$$

find location and order of the branch points, and give a valid branch cut.

3. Consider the function

$$f(z) = \sqrt{z^2+1} \ .$$

(a) Give location and order of the branch points of $f(z)$.

(b) Suppose evaluating f at the point $z = 2+2i$, then let z vary along the circle passing through $2+2i$ with centre at the origin, moving counterclockwise. When a full 2π cycle is completed by returning to the point $z = 2+2i$, determine whether or not f is restored to its initial value.

(c) Classify the behaviour of $f(z)$ at the point $z = \infty$.

(d) Describe a valid branch cut for $f(z)$ and the Riemann surface.

4. (a) Show that the inverse sine function $f(z) = \arcsin z$ is given by

$$f(z) = \arcsin z = \frac{1}{i}\ \ln(iz + \sqrt{1-z^2}) \ .$$

(b) Give the location and order of the branch points of this function.

(c) Consider the branch of $f(z) = \arcsin z$ defined using the branch cuts in Fig. 1, taking the principal branch of the logarithm and the branch of the square root such that $\sqrt{1-z^2} = 1$ when $z = 0$. Determine the value of $f(z)$ and of its derivative $f'(z)$ at the point $z = 3$.

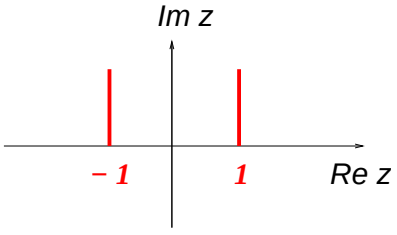


Fig. 1

5. Give the location and order of the branch points of the function

$$f(z) = \sqrt{z(z^2-1)}$$

and describe a valid branch cut.

6. Suppose that a branch of the function

$$f(z) = (z-1)^{2/3}$$

is defined by means of the branch cut in Fig. 2 and that it takes the value 1 when $z = 0$. Determine the value of $f(z)$ and of its derivative $f'(z)$ at the point $z = -i$.

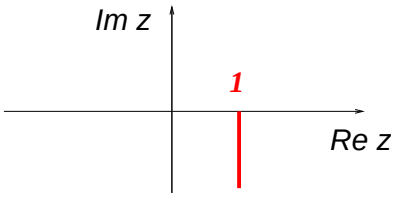


Fig. 2

II. Complex integration

7. (a) Calculate the integral

$$I = \int_L z^2 \, dz$$

where L is the straight-line segment in the complex z plane from point $z = 0$ to point $z = 2 + i$.

(b) Calculate the integrals

$$I_1 = \int_{L_1} z^2 \, dz \quad , \qquad I_2 = \int_{L_2} z^2 \, dz$$

where L_1 is the straight-line segment in the complex z plane from point $z = 0$ to point $z = 2$, and L_2 is the straight-line segment from point $z = 2$ to point $z = 2 + i$.

(c) Evaluate the difference of the integrals calculated above, $I - I_1 - I_2$, and interpret the result.

8. (a) Calculate the integral

$$I = \int_\gamma \overline{z} \, dz$$

where γ is the semicircle in the upper half z plane with centre at the origin and radius 1, traveled clockwise.

(b) Calculate the integral

$$I' = \int_{\gamma'} \overline{z} \, dz$$

where γ' is the semicircle in the lower half z plane with centre at the origin and radius 1, traveled counterclockwise.

(c) Evaluate the difference of the integrals calculated above, $I' - I$, and interpret the result.

9. Let I_n be the complex integral

$$I_n = \oint_{C_{a,r}} (z-a)^n \, dz$$

where $C_{a,r}$ is the circle of centre a and radius r, and n is an integer. Show by direct computation that $I_n = 0$ for $n \neq -1$, and $I_n = 2\pi i$ for $n = -1$.

10. Consider the integral of the function $f(z) = e^{iz^2}$ round the closed path Γ in the complex z plane given in Fig. 3:

$$\oint_\Gamma e^{iz^2}\, dz.$$

(a) Evaluate this integral for arbitrary $R > 0$ in Fig. 3.
(b) Consider the integral of f along the straight-line segment joining the points $z = Re^{i\pi/4}$ and

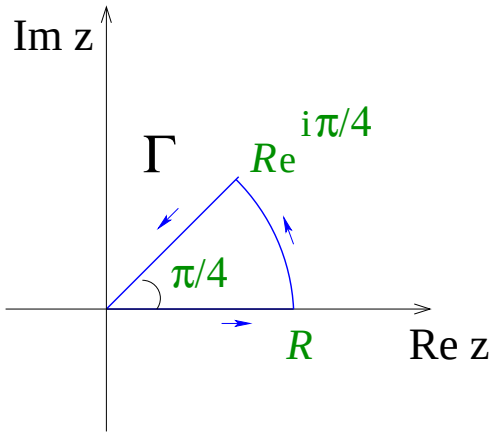


Fig.3

$z = 0$ in Fig. 3. Evaluate this integral for $R \to \infty$.
(c) Consider the integral of f along the circular arc in Fig. 3 between the points $z = R$ and $z = Re^{i\pi/4}$. Use the Darboux inequality to show that this integral vanishes for $R \to \infty$.
(d) Use the results in (a), (b), (c) to compute the real integrals

$$\int_0^\infty dx\ \cos x^2 \ , \quad \int_0^\infty dx\ \sin x^2 \quad \text{(Fresnel integrals)}.$$

11. Use Cauchy integral formulas to determine the value of

$$(a) \oint_\Gamma dz\ \frac{\cos z}{z(z^2+8)} \ , \qquad (b) \oint_\Gamma dz\ \frac{z}{2z+1} \ ,$$

where Γ is a square with centre at the origin and sides of length 2.

12. Use Cauchy integral formulas to determine the value of

$$(a) \oint_\gamma dz\ \frac{e^z}{z^3} \ , \qquad (b) \oint_\gamma dz\ \frac{\cosh z}{z^4} \ ,$$

where γ is the circle $|z-1| = 2$.

13. Show that if function f is holomorphic over the entire complex plane and is bounded (i.e., for any z, $|f(z)| \le M$ for a real constant M) then f must be constant.
[Consider Cauchy integral formula for the first derivative of f. Apply Darboux inequality to it.]

14. Apply Poisson integral formula to determine a function $u(x, y)$ that is harmonic in the upper half plane,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , \quad y > 0 \ ,$$

and obeys the following Dirichlet boundary condition on the real axis:

$$u(x, 0) = c(x) \ , \quad \text{with } c(x) = 1 \text{ for } x > 0 \ , \ c(x) = -1 \text{ for } x < 0 \ .$$

15. Use Gauss' mean value theorem to compute the following integrals:

$$(a)\ \int_0^{2\pi} \cos(\cos\theta + i\sin\theta)\ d\theta\ ,$$

$$(b)\ \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\ d\theta\ ,\quad (c)\ \int_0^{2\pi} \sin(\cos\theta)\sinh(\sin\theta)\ d\theta\ .$$

16. Verify that the following functions are harmonic

$$(a)\ f_1(x,y) = e^{-2xy}\sin(x^2-y^2) \qquad (b)\ f_2(x,y) = 2(1+x^2-y^2)+3x^2y-y^3$$

and determine their integral round the circle in the xy plane with centre at the origin and radius 1.

17. Take the principal branch of the square root function $f(z) = \sqrt{z}$

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}$$

defined with θ between 0 and 2π ($z = re^{i\theta}$) setting the branch cut from 0 to ∞ on the real positive semiaxis.
(a) Evaluate the integral of $\sqrt{z}$ on the circle $|z| = 1$.
[Suggestion: Consider the integral of $\sqrt{z}$ on the contour Γ in Fig. 4 and apply Cauchy theorem to this. Write down the relationship between the integral of $\sqrt{z}$ on Γ and the integral of $\sqrt{z}$ on the circle $|z| = 1$, taking the limit in which the radius ε of the inner circle in Fig. 4 goes to 0. Compute explicitly the contributions from the integrations along the straight-line segments above and below the branch cut.]

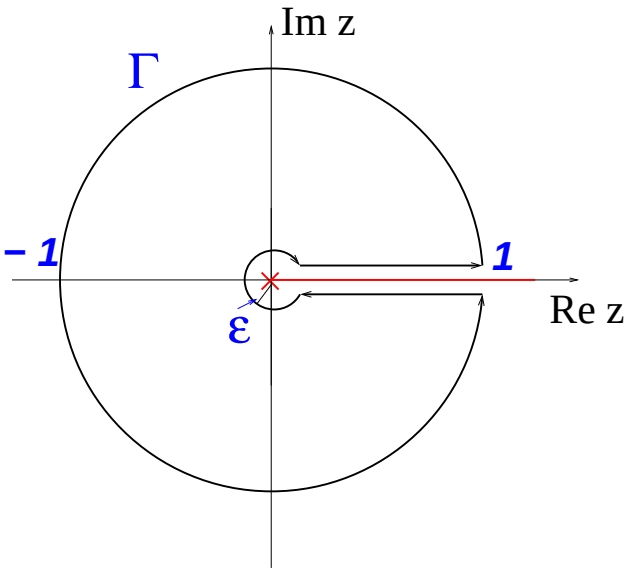


Fig.4

(b) Next consider the integral of $\sqrt{z}$ on any path in the upper half plane joining the points $z = -1$ and $z = 1$,

$$\int_{-1}^{1} \sqrt{z}\ dz\ ,$$

and the integral on any path in the lower half plane joining the same two points $z = -1$ and $z = 1$. Show that the results are different and are given respectively by $2(1+i)/3$ and $2(-1+i)/3$. Use this as a cross-check on the result of part (a).

Functions of a complex variable (S1)

Problem sheet 3

I. Power series expansions; singular points

1. (a) Represent the function $f(z) = (z+1)/(z-1)$ in Taylor series about $z = 0$ and determine the region of convergence. (b) Represent f in Laurent series about $z = 0$ for $|z| > 1$.

2. Find the Laurent series expansion for the function $f(z) = 1/[(z+1)(z+3)]$
(a) in the disk $|z| < 1$; (b) in the annulus $1 < |z| < 3$; (c) in the region $|z| > 3$.
(d) Write the Laurent series for $f(z)$ about the point $z = -1$ and give its region of convergence.

3. Find the Laurent series for $f(z) = z^{-2}(1-z)^{-1}$ in the regions (a) $0 < |z| < 1$; (b) $|z| > 1$.

4. Expand the function $z/(1+z^3)$ in power series of z valid in the regions (a) $|z| < 1$; (b) $|z| > 1$.

5. Determine the first four terms of the Laurent series expansion of $f(z) = e^z/[z(z^2+1)]$ valid for $0 < |z| < 1$.

6. Determine the first four terms of the Laurent series expansion of $f(z) = (z-3)\sin[1/(z+2)]$ about the point $z = -2$, and give the region of convergence of the series.

7. Determine the first four terms of the Laurent series expansion of $f(z) = [z(z-3)]^{-2}$ about the point $z = 3$, and give the region of convergence of the series.

8. (a) Locate and classify the singular points of $f(z) = 1/[z^2(1+e^{1/z})]$. (b) Does f have a Laurent series expansion about $z = 0$?

9. Determine the behavior at $z = \infty$ for the functions (a) z^2; (b) e^{-z}; (c) e^{-1/z^2}; (d) $\tan z$.

10. For each of the following functions,

$$\text{i)}\ \frac{1}{z^2\ \sinh z}\ , \qquad \text{ii)}\ ze^{1/z^2}\ , \qquad \text{iii)}\ \frac{\cos z}{z^5}\ , \qquad \text{iv)}\ \cosh\frac{1}{z}\ ,$$

(a) obtain the Laurent series about $z = 0$ and give the region of convergence; (b) classify the singularity at $z = 0$; (c) evaluate the integral of the function round the circle $|z| = 1$.

II. Residue calculus

11. Locate and classify the singular points in the complex z plane for each of the following functions,

$$\text{(a)}\ \frac{1-z}{(1-2z)^2}\ , \qquad \text{(b)}\ e^{1/z^2}\ , \qquad \text{(c)}\ \cot z\ ,$$

and determine the residue of the function at the singularity.

12. Calculate the following contour integrals in the complex plane:

$$(a)\ \oint_{|z|=2} \frac{3z+1}{z(z-1)^3}\,dz\ , \qquad (b)\ \oint_{|z|=3/2} \frac{1-z^2}{1+z^2}\,\frac{dz}{z}\ , \qquad (c)\ \oint_{|z-1|=3/2} \frac{e^{1/z}}{z^2-1}\,dz\ .$$

13. Calculate the following real integrals

$$(a)\ \int_0^\infty \frac{x^2}{(x^2+1)\,(x^2+4)}\,dx\ , \qquad (b)\ \int_0^\infty \frac{\cos 3x}{1+x^2}\,dx\ , \qquad (c)\ \int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\,d\theta$$

by complex contour integration methods.

14. Calculate the integral along the real axis

$$\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{1+x^2}\,dx \qquad (\lambda \in \mathbb{R})$$

by complex contour integration. [Refer to Fig. 1. Discuss the contour in the complex z plane with relation to the sign of λ, based on the behavior of the exponential on the semicircular arc.]

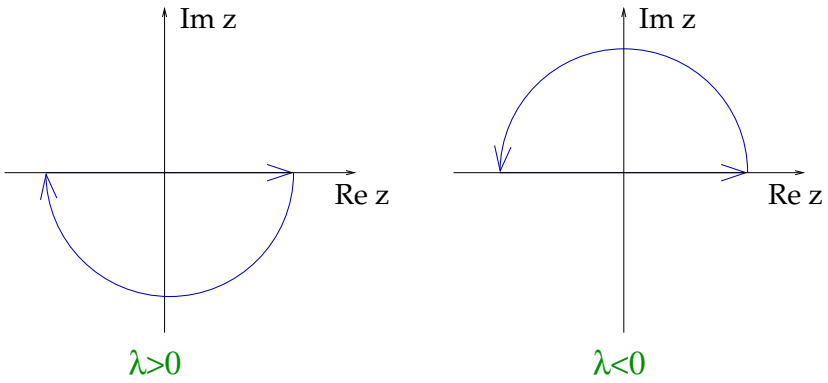


Fig.1

15. Calculate the following real integrals

$$(a)\ \int_0^{2\pi} e^{\cos\theta}\,\cos(\theta-\sin\theta)\,d\theta\ , \qquad (b)\ \int_0^{2\pi} e^{\cos\theta}\,\sin(\theta-\sin\theta)\,d\theta\ ,$$

$$(c)\ \int_0^{2\pi} e^{-\cos\theta}\,\cos(\theta+\sin\theta)\,d\theta\ , \qquad (d)\ \int_0^{2\pi} e^{-\cos\theta}\,\sin(\theta+\sin\theta)\,d\theta\ ,$$

by complex contour integration methods. [Suggestion. Consider the integral of $e^{1/z}$ and of $e^{-1/z}$ on the unit circle centered at the origin in the complex z plane. Evaluate these integrals by residue theorem. Relate their real and imaginary parts to the given integrals.]

16. Apply complex contour integration methods to compute

$$I = \int_{-\infty}^{+\infty} \frac{e^{x/2}}{\cosh x}\,dx\ .$$

[Suggestion. Evaluate the integral of the complex-valued function $f(z) = \exp(z/2)/\cosh z$ along the rectangular contour R in the complex z plane depicted in Fig. 2. Relate this result to the given integral I for $L \to \infty$.]

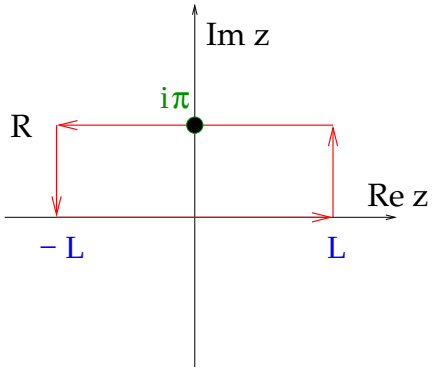


Fig.2

17. (a) Apply complex integration methods to compute the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad .$$

[Suggestion. Consider the integral of the complex-valued function $f(z) = \pi \cot(\pi z)/z^2$ along the square contour Q_N in the complex z plane depicted in Fig. 3, where N is a natural number ≥ 1. Evaluate this integral using residue theorem. Use the result to compute the sum of the given series, by examining the limit $N \to \infty$.]

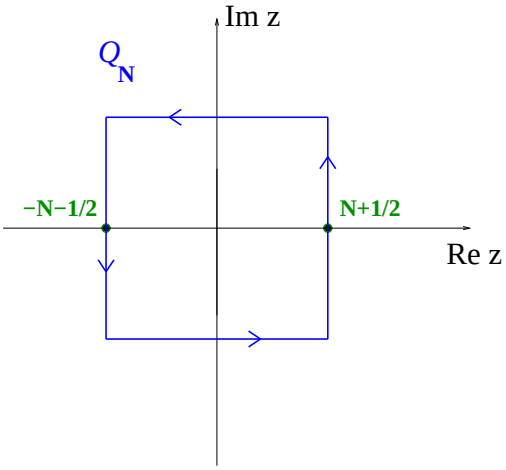


Fig.3

(b) Extend the above calculation to compute the sum of the series

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \qquad (a \in \mathbb{R}) \quad .$$

18. (a) Take the principal branch of the logarithm function $\ln z$ and evaluate the integral

$$\oint_{\Gamma} dz \, \frac{(\ln z)^2}{z^2 + 1} \quad ,$$

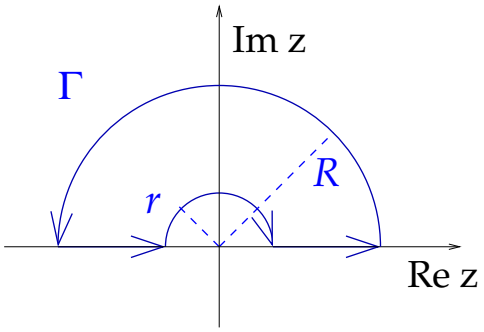


Fig.4

where Γ is the closed contour in Fig. 4, consisting of two semicircles in the upper half plane with centre at the origin and radii r and R respectively ($r < 1$, $R > 1$), and intervals $(-R, -r)$ and (r, R) on the real axis.

(b) Use the result in (a) to calculate the real-axis integrals

$$i) \int_0^\infty dx \, \frac{\ln x}{x^2+1} \quad , \qquad ii) \int_0^\infty dx \, \frac{(\ln x)^2}{x^2+1} \quad .$$

19. Take the principal branch of the function

$$f(z) = \frac{1}{\sqrt{z}}$$

defined by setting the branch cut along the negative real semiaxis. Calculate the integral in the complex plane

$$\int_\gamma e^z \, \frac{1}{\sqrt{z}} \, dz \, ,$$

where γ is the straight line parallel to the imaginary axis with real part equal to 1.
[Suggestion. Consider the integral round the closed contour Γ in Fig. 5. Apply Cauchy theorem to this. Let the radii of the small circle and of the large circle in Fig. 5 tend to 0 and ∞ respectively, and apply Jordan lemma. Obtain the result by evaluating the integral along the branch cut.]

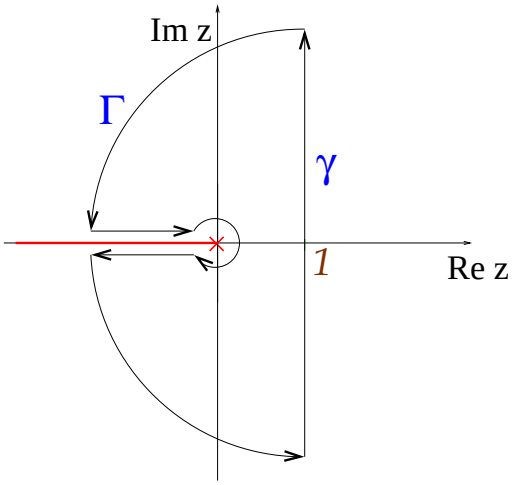


Fig.5

Functions of a complex variable (S1)

Problem sheet 4

I. Residue calculus (part 2)

1. (a) Use complex contour integration to compute

$$I = \int_0^\infty \frac{1}{1+x^3}\, dx \ .$$

[Hint: Evaluate the integral of the complex-valued function $f(z) = 1/(1+z^3)$ round the contour Γ in Fig. 1 using residue theorem. Relate this result for $R \to \infty$ to the given integral I.]

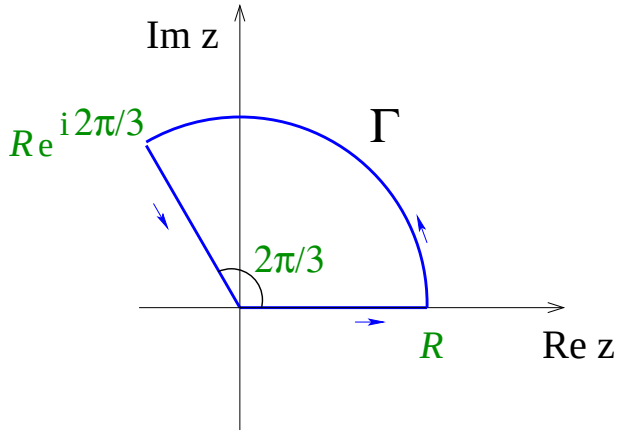


Fig.1

(b) Generalize the computation in (a) to calculate

$$I_n = \int_0^\infty \frac{1}{1+x^n}\, dx$$

for any $n \geq 2$. [Modify the contour in Fig. 1 so that the circular arc goes from R to $Re^{2\pi i/n}$]

2. (a) Use complex contour integration to compute

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx \ .$$

[Suggestion: Integrate the complex-valued function $f(z) = \exp(iz)/z$ round the contour Γ in the complex plane depicted in Fig. 2. Evaluate this integral using Cauchy theorem, then relate it to the given real-variable integral. Observe that while the contribution from the semicircle of radius R in Fig. 2 vanishes for $R \to \infty$, the contribution from the semicircle of radius r is non-vanishing for $r \to 0$.]

(b) Apply similar methods to compute

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx \ .$$

[Suggestion: Take the complex-valued function $f(z) = [\exp(2iz) - 1]/z^2$ and integrate along the same contour Γ as in Fig. 2.]

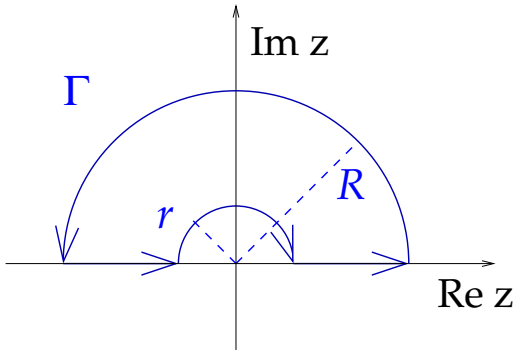


Fig.2

3. Calculate the integral of the functions

$$(a)\ \frac{\sin z}{z^4}\ , \qquad (b)\ \frac{1}{z^4 \sin z}\ , \qquad (c)\ \frac{\cos z}{z^4}\ , \qquad (d)\ \frac{\tan z}{z^4}$$

round the circle with centre at the origin and radius 1 in the complex z plane.

4. Let C be the circle $|z| = 4$ in the complex z plane. Consider the functions

$$(a)\ f(z) = \frac{z+1}{z}\ , \qquad (b)\ f(z) = \frac{1}{z^2}\ .$$

For each of these functions f, determine the winding number of the image of C through f about the origin

$$J = \frac{1}{2\pi}\ \Delta_C\ \arg f(z)\ .$$

Then determine the number of poles P and number of zeros N of f inside C, including their multiplicity, and verify the argument principle, $J = N - P$.

5. Determine the number of roots of the equations

$$(a)\ \ z^7 - 4z^3 + z - 1 = 0\ , \qquad (b)\ \ 3z^6 = e^z$$

in the disk $|z| < 1$.

6. Determine the number of roots of the equation

$$2 + z^2 - e^{iz} = 0$$

in the upper half plane.

7. Calculate the sum of the following series

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\ , \qquad (b)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

using complex contour integration methods.

8. Apply complex-plane techniques for integration in the presence of branch cuts to evaluate the following real integrals:

$$(a)\ \int_0^{\infty} \frac{\sqrt{x}}{x^2+1}\ dx\ , \qquad (b)\ \int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+x}\ dx\ .$$

9. Apply complex-plane techniques for integration in the presence of branch cuts to evaluate the following real integrals:

$$(a)\ \int_0^{\infty} \frac{\ln x}{(1+x^2)^2}\ dx\ , \qquad (b)\ \int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2}\ dx\ .$$

II. Integral transforms

10. Calculate the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega - i\varepsilon}\,d\omega$$

for real negative t and for real positive t, with ε a positive real constant. Verify that this provides an integral representation for the step function $\Theta(t)$ [$\Theta(t) = 1$ for $t > 0$, $\Theta(t) = 0$ for $t < 0$].

11. Consider the rectangular pulse in Fig. 3a

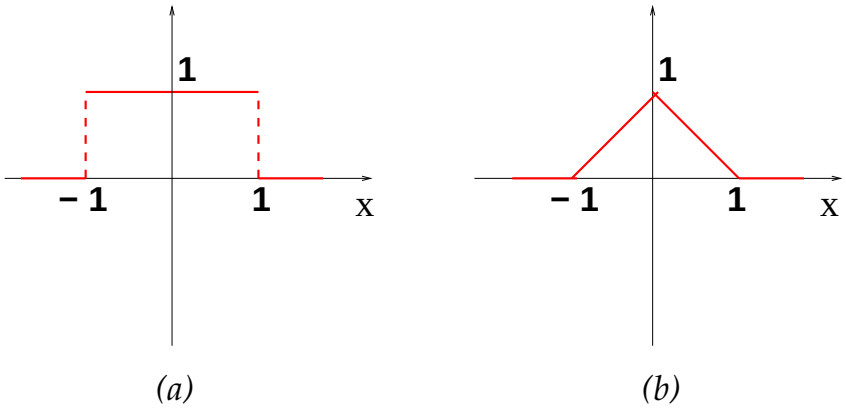
$$(a)\;\; f(x) = \Theta(1 - |x|)$$

and the triangular pulse in Fig. 3b

$$(b)\;\; f(x) = (1 - |x|)\;\Theta(1 - |x|)\;\;.$$

In each case, evaluate the Fourier transform

$$\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} dx\; e^{-i\omega x}\; f(x)\;\;.$$



(a) *(b)*

Fig.3

12. Calculate the Laplace transform

$$F(z) = \int_{0}^{+\infty} dt\; e^{-zt}\; f(t)$$

of the functions

$$(a)\;\; f(t) = \sin t\;\;, \qquad (b)\;\; f(t) = t\sin t\;\;, \qquad (c)\;\; f(t) = \cosh t\;\;.$$

13. Take the function

$$F(z) = \frac{1}{\sqrt{z}}\;\;.$$

Calculate the integral in the complex z plane

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt}\; F(z)\; dz \qquad (a > 0)\;\;,$$

which defines the inverse Laplace transform of $F(z)$. The integral runs along the straight line parallel to the imaginary axis and with real part a ($a > 0$). [Hint: Evaluate the integral round the closed path Γ in Fig. 4 by residue theorem, and relate it to the integral that defines $f(t)$.]

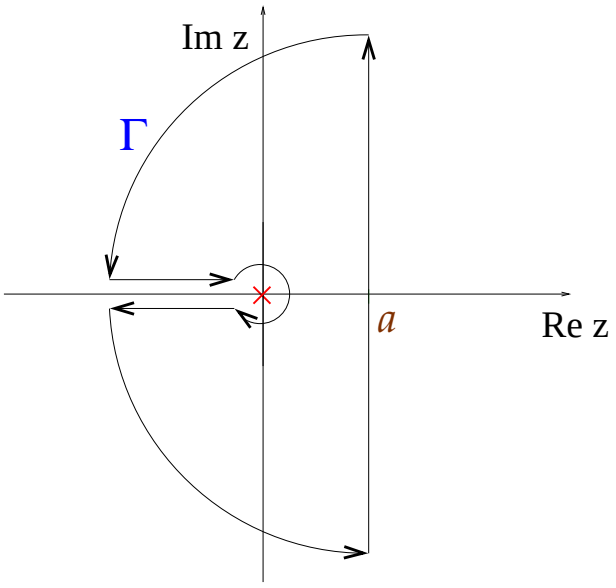


Fig.4

14. Suppose the function $u(x,t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}$$

for $-\infty < x < \infty$ and $t \geq 0$, with the initial-value condition

$$u(x,0) = h(x) \quad ,$$

where λ is a real positive constant (diffusion coefficient) and $h(x)$ is a given function of x. Consider the Fourier transform of u with respect to x

$$\widetilde{u}(p,t) = \int_{-\infty}^{+\infty} dx \; e^{-ipx} \; u(x,t) \quad .$$

(a) Obtain the differential equation for $\widetilde{u}(p,t)$ and the corresponding initial-value condition.
(b) Write the solution of the initial-value problem for $\widetilde{u}(p,t)$.
(c) Determine the solution $u(x,t)$ of the starting equation as a Fourier convolution integral over the initial distribution $h(x)$.

15. Apply the Laplace transformation to determine the function $y(t)$ satisfying the equation

$$y(t) + \int_0^t dt' \; (t-t') \; y(t') = \sin 2t \quad , \qquad t > 0 \quad .$$

(a) First, Laplace-transform the equation and show that the second term on the left hand side of the equation gives $\widetilde{y}(z)/z^2$, where $\widetilde{y}$ is the Laplace transform of y,

$$\widetilde{y}(z) = \int_0^{+\infty} dt \; e^{-zt} \; y(t) \quad .$$

Determine the explicit expression for $\widetilde{y}(z)$.
(b) Next, find the solution for the function $y(t)$ by evaluating the inverse Laplace transform of $\widetilde{y}(z)$.

Functions of a complex variable (S1)

Answers for Problem Sheet 1

1. 1, $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$.

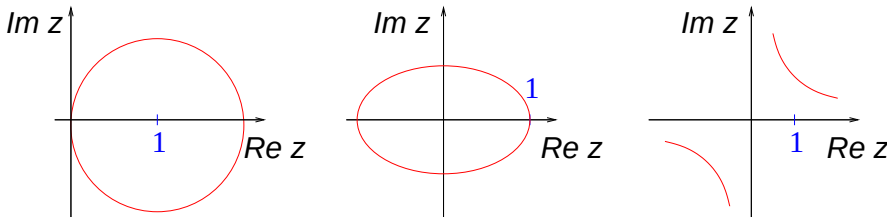
2. $2^{1/6}e^{i\pi/4}$, $2^{1/6}e^{i11\pi/12}$, $2^{1/6}e^{i19\pi/12}$.

3. $z = \cot[\pi(1+2n)/10]$, $n = 0, 1, 2, 3, 4$.

4. (a) $\mathrm{Re}\cos i = (e + 1/e)/2$, $\mathrm{Im}\cos i = 0$; (b) $\mathrm{Re}\sin i = 0$, $\mathrm{Im}\sin i = (e - 1/e)/2$.

6. (a) $z = i\pi(2n+1)$ (b) $z = \ln 2 + i\pi(2n+1)$
(c) $z = in\pi$ (d) $z = \pi(2n + 1/2) - i\ln|3 \pm 2\sqrt{2}|$ $(n \in \mathbb{Z})$

7.



(a) (b) (c)

8. (a) $e^{-\pi/2-2\pi n}$ (b) $e^{-\pi/4-2\pi n}[\cos(2^{-1}\ln 2) + i\sin(2^{-1}\ln 2)]$ (c) $e^{i(1+2n)}$ $(n \in \mathbb{Z})$

9. (a) $2^{-1}\ln 2 + 7\pi i/4$ (b) $2^{-1}\ln 2 + \pi i/4$ (c) $i\pi/2$

10. (a) $z = 0$. (b) no interior points; $z = 0$ and all points in S are boundary points.
(c) not open; not closed; bounded; not connected; not compact.

12. (a) holomorphic in $\mathbb{C} \setminus \{z = 2n\pi i\ ,\ n \in \mathbb{Z}\}$; $f'(z) = -e^z/(e^z - 1)^2$
(b) holomorphic in $\mathbb{C} \setminus \{z = i\ ,\ z = -i\}$; $f'(z) = -4z/(1 + z^2)^3$
(c) nowhere holomorphic
(d) holomorphic in $\mathbb{C} \setminus \{z = \pi(n + 1/2)\ ,\ n \in \mathbb{Z}\}$; $f'(z) = 1/\cos^2 z$

14. (a) (i) harmonic; (ii) harmonic; (iii) harmonic; (iv) not harmonic; (v) harmonic
(b) (i) $u(x, y) = x^2 - y^2 - x$ real part of $f(z) = z^2 - z$; harmonic conjugate $v(x, y) = 2xy - y$.
(ii) $u(x, y) = \sin x \cosh y$ real part of $f(z) = \sin z$; harmonic conjugate $v(x, y) = \cos x \sinh y$.
(iii) $u(x, y) = e^{-x}\cos y + xy$ real part of $f(z) = e^{-z} - iz^2/2$; harmonic conjugate $v(x, y)$ $= (y^2 - x^2)/2 - e^{-x}\sin y$.
(v) $u(x, y) = x - y$ real part of $f(z) = z(1 + i)$; harmonic conjugate $v(x, y) = x + y$.

16. (a) $f(z) = ize^{-z}$; (b) $e^{-x}(x\sin y - y\cos y) = \text{const}$.

17. (a) $z = \infty$ essential singularity, $z = -i$ pole (isolated); (b) $z = 1, -1, 2i, -2i$ isolated poles;
(c) $z = 1/(n\pi)$, $n = \pm 1, \pm 2, \ldots$ isolated poles; $z = 0$ non-isolated singular point; $z = \infty$ pole.

18. (a) pole; (b) essential singularity; (c) holomorphic

19. (a) f holomorphic in the complex plane except at 0 and ∞.
(b) $f'(z) = 1 - 1/z^2 \Rightarrow$ conformal except at points $z = +1$ and $z = -1$.
(c) upper and lower unit semicircles mapped onto $[-2, 2]$.
(d) ellipse in uv plane $u^2/(\rho + 1/\rho)^2 + v^2/(\rho - 1/\rho)^2 = 1$.

20. (a) $f'(z) = -2i/(i + z)^2 \neq 0 \Rightarrow$ conformal; (b) circle $|w| = 1$; (c) disk $|w| < 1$.

Functions of a complex variable (S1)

Answers for Problem Sheet 2

1. (a) 1 and ∞ are 2nd-order branch points; 1 to $+\infty$ on real axis is valid branch cut.
   (b) 3-sheeted, closed surface; three sheets R_0, R_1, R_2 joined along cut $(1, +\infty)$;
   lower edge of cut in R_2 joined back to upper edge of cut in R_0;
   images of 3 sheets are $0 \le \arg w \le 2\pi/3$; $2\pi/3 \le \arg w \le 4\pi/3$; $4\pi/3 \le \arg w \le 2\pi$.

2. (a) 1 and -1 are ∞-order branch points; 1 to -1 on real axis is valid branch cut.
   (b) $-i$ and ∞ are ∞-order branch points; $-i$ to $-i\infty$ on imaginary axis is valid branch cut.
   (c) 1, -1 and ∞ are ∞-order branch points; $-\infty$ to -1 and 1 to $+\infty$ on real axis is valid branch cut.

3. (a) i and $-i$ are 1st-order branch points; (b) f restored to initial value;
   (c) $z = \infty$ simple pole (no branch point); (d) The segment $-i$ to i on imaginary axis is valid branch cut. The Riemann surface is closed, made of two sheets joined along the cut; edges on opposite sides of cut from the two sheets are joined together.
   $-i\infty$ to $-i$ and i to $+i\infty$ is also a valid branch cut.

4. (b) 1 and -1 are 1st-order branch points; ∞ is ∞-order branch point;
   (c) $f(3) = \pi/2 - i\ln(3 + 2\sqrt{2})$; $f'(3) = -i/\sqrt{8}$.

5. 1, -1, 0, ∞ are 1st-order branch points; -1 to 0 and 1 to $+\infty$ on real axis is valid branch cut.

6. $f(-i) = 2^{1/3}(\sqrt{3}/2 + i/2)$, $f'(-i) = -2^{5/6}e^{-i\pi/12}/3$.

7. (a) $I = (2 + 11i)/3$ (b) $I_1 = 8/3$, $I_2 = -2 + 11i/3$
   (c) $I - I_1 - I_2 = 0$, embodying Cauchy theorem (z^2 holomorphic). $\Rightarrow$ I obtainable from primitive function $(z^3/3)|_0^{2+i}$.

8. (a) $I = -i\pi$ (b) $I' = i\pi$
   (c) $I' - I = 2\pi i \neq 0$ ($\overline{z}$ not holomorphic). On circle $|z| = 1$, $\overline{z} = 1/z \Rightarrow I' - I$ must equal $\int_{|z|=1} dz/z = 2\pi i$.

10. (a) 0 (b) $-e^{i\pi/4}\sqrt{\pi}/2$ (d) $\sqrt{\pi}/(2\sqrt{2})$, $\sqrt{\pi}/(2\sqrt{2})$

11. (a) $i\pi/4$ (b) $-i\pi/2$

12. (a) $i\pi$ (b) 0

14. $(2/\pi)\arctan(x/y)$

15. (a) 2π (b) 2π (c) 0

16. (a) 0 (b) 4π

17. (a) $-4/3$

Functions of a complex variable (S1)

Answers for Problem Sheet 3

1. (a) $-1-2\sum_{n=1}^{+\infty} z^n$, $|z|<1$ (b) $1+2\sum_{n=1}^{+\infty} z^{-n}$

2. (a) $\sum_{n=0}^{+\infty}(-1)^n[(1-3^{-n-1})/2]z^n$ (b) $-1/6-(1/2)[\sum_{n=1}^{+\infty}(-1)^n 3^{-n-1}z^n+\sum_{n=1}^{+\infty}(-1)^n z^{-n}]$
   (c) $(1/2)\sum_{n=2}^{+\infty}(-1)^n(3^{n-1}-1)z^{-n}$ (d) $\sum_{n=0}^{+\infty}(-1)^n 2^{-n-1}(z+1)^{n-1}$, $0<|z+1|<2$

3. (a) $\sum_{n=0}^{+\infty} z^{n-2}$ (b) $-\sum_{n=0}^{+\infty} z^{-n-3}$

4. (a) $\sum_{n=0}^{+\infty}(-1)^n z^{3n+1}$ (b) $\sum_{n=0}^{+\infty}(-1)^n z^{-3n-2}$

5. $1/z+1-z/2-5z^2/6+\ldots$

6. $1-5(z+2)^{-1}-(1/6)(z+2)^{-2}+(5/6)(z+2)^{-3}+\ldots$, $|z+2|>0$

7. $(1/9)(z-3)^{-2}-(2/27)(z-3)^{-1}+(1/27)-(4/243)(z-3)+\ldots$, $0<|z-3|<3$

8. (a) $z=i/[\pi(2n+1)]$, $n=0,\pm1,\ldots$: simple poles; $z=0$: non-isolated singularity (b) no

9. (a) pole of order 2 (b) essential singularity
   (c) holomorphic (d) non-isolated singularity

10. i): (a) $1/z^3-1/(6z)+7z/360+\ldots$, $0<|z|<\pi$ (b) pole of order 3 (c) $-i\pi/3$
    ii): (a) $z+1/z+1/(2z^3)+\ldots$, $|z|>0$ (b) essential singularity (c) $2\pi i$
    iii): (a) $1/z^5-1/(2z^3)+1/(24z)-z/720+\ldots$, $|z|>0$ (b) pole of order 5 (c) $i\pi/12$
    iv): (a) $1+1/(2z^2)+1/(24z^4)+\ldots$, $|z|>0$ (b) essential singularity (c) 0

11. (a) $z=1/2$ pole of order 2; $\mathrm{Res}_{z=1/2}\ \mathrm{f}=-1/4$
    (b) $z=0$ essential singularity; $\mathrm{Res}_{z=0}\ \mathrm{f}=0$
    (c) $z=n\pi$ ($n=0,\pm1,\ldots$) poles of order 1; $\mathrm{Res}_{z=n\pi}\ \mathrm{f}=1$

12. (a) 0 (b) $-2\pi i$ (c) $i\pi/e$

13. (a) $\pi/6$ (b) $\pi/(2e^3)$ (c) $2\pi/3$

14. $\pi\ e^{-|\lambda|}$

15. (a) 2π (b) 0 (c) -2π (d) 0

16. $\pi\sqrt{2}$

17. (a) $\pi^2/6$ (b) $(\pi a\ \coth\pi a-1)/(2a^2)$

18. (a) $-\pi^3/4$ (b) *i)* 0 *ii)* $\pi^3/8$

19. $2i\sqrt{\pi}$

Functions of a complex variable (S1)

Answers for Problem Sheet 4

1. (a) $2\pi/(3\sqrt{3})$ (b) $(\pi/n)/\sin(\pi/n)$

2. (a) π (b) π

3. (a) $-i\pi/3$ (b) $7\pi i/180$ (c) 0 (d) $2\pi i/3$

4. (a) $J = 0$; $P = 1$, $N = 1$ (b) $J = -2$; $P = 2$, $N = 0$

5. (a) 3 (b) 6

6. 1

7. (a) $\pi^2/12$ (b) $\pi^3/32$

8. (a) $\pi/\sqrt{2}$ (b) $2\pi/\sqrt{3}$

9. (a) $-\pi/4$ (b) $\pi \ln 2$

10. 1 for $t > 0$; 0 for $t < 0$

11. (a) $\widetilde{f}(\omega) = 2\sin\omega/\omega$ (b) $\widetilde{f}(\omega) = \sin^2(\omega/2)/(\omega/2)^2$

12. (a) $F(z) = 1/(1+z^2)$ (b) $F(z) = 2z/(1+z^2)^2$ (c) $F(z) = z/(z^2-1)$

13. $1/\sqrt{\pi t}$

14. (a) $\partial\widetilde{u}/\partial t = -\lambda p^2 \widetilde{u}$, $\widetilde{u}(p,0) = \widetilde{h}(p)$
    (b) $\widetilde{u}(p,t) = \widetilde{h}(p)\ e^{-\lambda p^2 t}$
    (c) $u(x,t) = (1/\sqrt{4\pi\lambda t}) \int_{-\infty}^{\infty} dy\ e^{-(x-y)^2/(4\lambda t)} h(y)$

15. (a) $\widetilde{y} + \widetilde{y}/z^2 = 2/(z^2+4)$; $\widetilde{y}(z) = 2z^2/[(z^2+1)(z^2+4)]$
    (b) $y(t) = (4/3)\sin 2t - (2/3)\sin t$