Complex numbers

john.magorrian@physics.ox.ac.uk

Three lectures:

1. Fundamental ideas (revision)
2. Some important functions of a complex variable
3. Applications

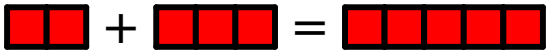
Numbers: a recap

A long time ago (in a classroom far far away?)
you learned how to count!

You learned how

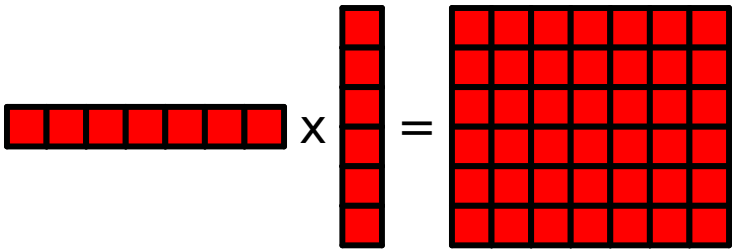
- to add whole numbers:

$$2 + 3 = 5;$$



- and how to multiply them:

$$7 \times 6 = 42.$$



Beyond the whole numbers

You were introduced to the invisible number 0.

Around the same time you learned how to undo addition:

$$2 + 3 = 5,$$

$$5 - 3 = 2.$$

That had an interesting consequence:

$$2 - 3 = ??$$

Whole numbers aren't the whole story! **Integers** $\mathbb{Z} = \{..., -2, -1, 0, +1, +2, ...\}$.

Beyond the whole numbers

You were introduced to the invisible number 0.

Around the same time you learned how to undo addition:

$$2 + 3 = 5,$$

$$5 - 3 = 2.$$

That had an interesting consequence:

$$2 - 3 = ??$$

Whole numbers aren't the whole story! **Integers** $\mathbb{Z} = \{..., -2, -1, 0, +1, +2, ...\}$.

When you learned how to undo multiplication, there was another new idea:

$$6 \times 7 = 42$$

$$42 \div 7 = 6$$

$$6 \div 7 = ??$$

Enter the set of **rational numbers,** $\mathbb{Q} \equiv \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z}, \ q \neq 0 \right\}$.

Unlike $\mathbb{Z}$, $\mathbb{Q}$ is "complete" under $+ \ - \ \times \ \div$.

From arithmetic to al jabr jabr

All this was warming you up for the joy of x.

You learned how to find the x that satisfied beasts like

$$7x + 11 = -3.$$

You could even invite y and z along to join in the fun –

$$\begin{aligned} x + y \quad &= 0, \\ 2x - y + z &= 1, \\ y + 3z &= 4 \end{aligned}$$

– secure in the knowledge that they too would be rational.

Maths made sense. It was rational.

Irrational numbers

Let's try to find a rational solution to $x^2 = 2$.

Assumption: suppose that
- $x = p/q$, where $p, q \in \mathbb{Z}$, and
- p and q share no common factor (Euclid's algorithm).

Then $p^2/q^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even.

Now substitute $p = 2r$ $(r \in \mathbb{Z})$ into $x^2 = 2$:

$(2r)^2/q^2 = 2 \Rightarrow 2r^2 = q^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even.

Both p and q are divisible by 2!

This contradicts our assumption above.

Resolution: assumption is false; $x \notin \mathbb{Q}$!

The real numbers

NB minus sign! [Someone miscounted (-1)s during the lecture...]

$$\sqrt{2} = 2\left(1 - \frac{1}{2}\right)^{1/2} = 2\left(1 - \frac{1}{4} - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{2^{2k}k!}\right)$$

is a series of numbers $\in \mathbb{Q}$ whose limit $\notin \mathbb{Q}$. The set $\mathbb{R}$ of real numbers "completes" $\mathbb{Q}$ to include such limits.

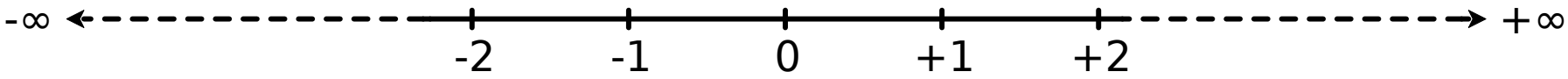
The real numbers

NB minus sign! [Someone miscounted (-1)s during the lecture...]

$$\sqrt{2} = 2\left(1 - \frac{1}{2}\right)^{1/2} = 2\left(1 - \frac{1}{4} - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{2^{2k}k!}\right)$$

is a series of numbers $\in \mathbb{Q}$ whose limit $\notin \mathbb{Q}$. The set $\mathbb{R}$ of real numbers "completes" $\mathbb{Q}$ to include such limits.

Think of real numbers as being represented by points along an infinite line – the real line – along which integers are evenly spaced.



Given any real number we can always find a rational one that approximates it arbitrarily closely:

$$\sqrt{2} = 1.414213562373095\ldots \simeq \frac{1414213562373}{1000000000000}.$$

Exercise: Show that the decimal expansion of $x \in \mathbb{R}$ repeats iff $x \in \mathbb{Q}$. For example, $\frac{22}{7} = 3.\underbrace{142857}\underbrace{142857}142\ldots$ repeats with period 6.

Consequence: "most" reals aren't rational!

Solving quadratic equations

No rational number will solve $x^2 = 2$. What about more general quadratic equations?

Completing the square:

$$\boxed{ax^2 + bx = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}.}$$

Let's use this to find the general solution of

$$\begin{aligned} 0 &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \end{aligned}$$

Rearranging,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\sqrt{}$ in this expression means $x \notin \mathbb{Q}$ in general. But what if $b^2 - 4ac < 0$?

Imaginary numbers, complex numbers

Consider the solution $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to $az^2 + bz + c = 0$. What happens if $b^2 - 4ac < 0$?

Introduce set of imaginary numbers $\{x \times \mathrm{i} \,|\, x \in \mathbb{R}\}$, where $\mathrm{i} \equiv \sqrt{-1}$.

Adding two imaginary numbers or multiplying one by a real produces another imaginary number, but multiplying two imaginary numbers produces a real.

Imaginary numbers, complex numbers

Consider the solution $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to $az^2 + bz + c = 0$. What happens if $b^2 - 4ac < 0$?

Introduce set of imaginary numbers $\{x \times \mathrm{i} \,|\, x \in \mathbb{R}\}$, where $\mathrm{i} \equiv \sqrt{-1}$.

Adding two imaginary numbers or multiplying one by a real produces another imaginary number, but multiplying two imaginary numbers produces a real.

Adding a real number to an imaginary one produces a complex number. For example, $z = 4 + 3\mathrm{i}$. Each such number has a real and an imaginary part:

$$\mathrm{Re}\, z = 4; \quad \mathrm{Im}\, z = 3. \qquad [\text{NB } \mathrm{Im}\, z \in \mathbb{R} \text{ always!}]$$

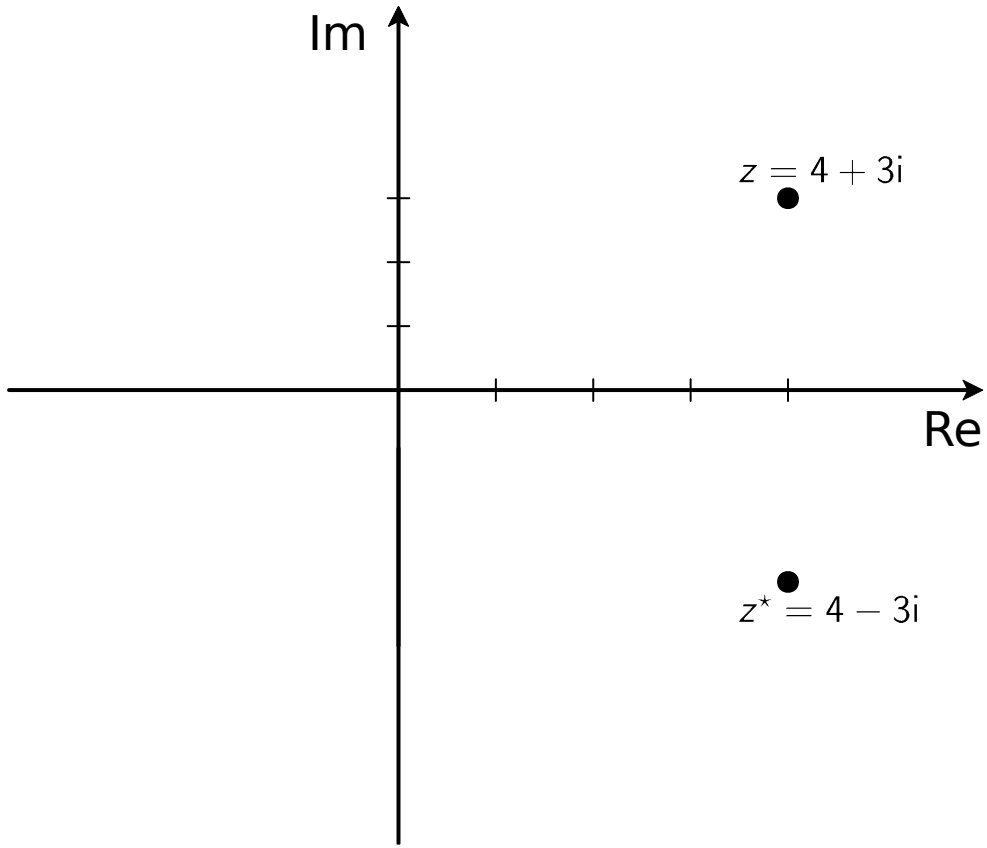
Two complex numbers are equal if both real and imaginary parts agree.

Complex numbers obey the usual rules of arithmetic with $\mathrm{i} \equiv \sqrt{-1}$ as a magic constant, for which $\mathrm{i}^2 = -1$. Use $\mathbb{C}$ to denote them.

Complex numbers

Complex numbers inhabit the complex plane.

Every $z \in \mathbb{C}$ has a complex conjugate $z^\star$ obtained by reflecting z about the Re axis: $\operatorname{Im} z^\star = -\operatorname{Im} z$. Real numbers have $z = z^\star$.



Complex numbers

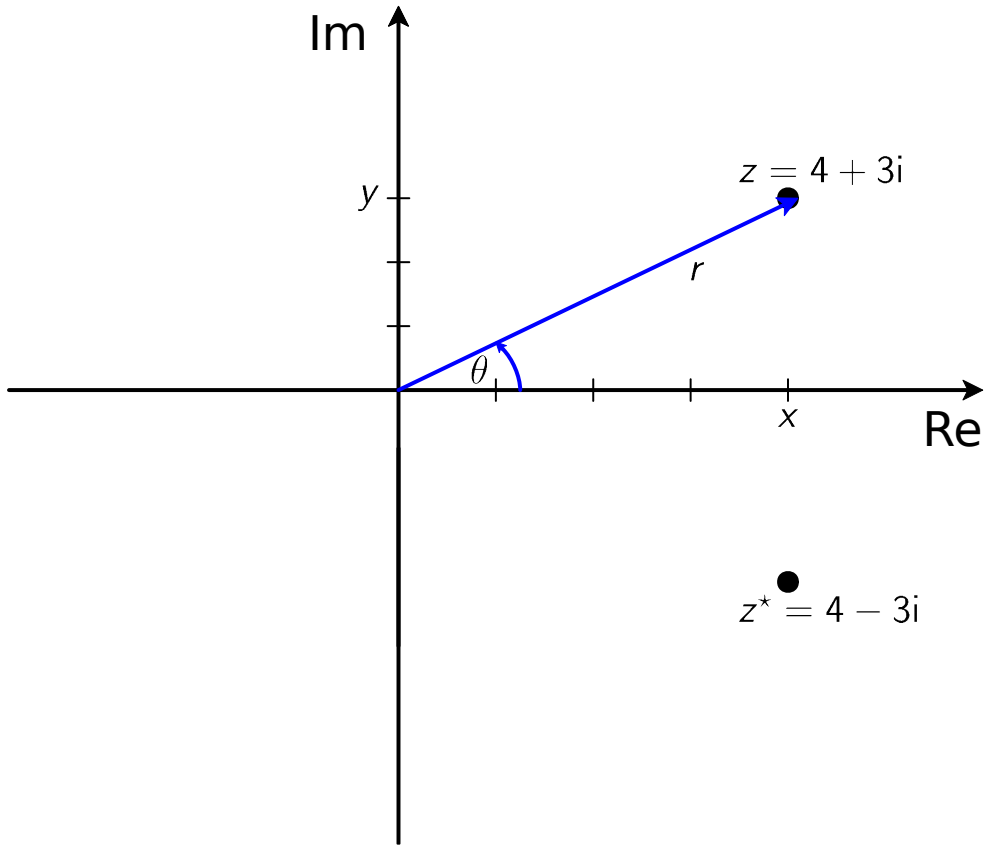
Complex numbers inhabit the complex plane.

Every $z \in \mathbb{C}$ has a complex conjugate $z^\star$ obtained by reflecting z about the Re axis: $\operatorname{Im} z^\star = -\operatorname{Im} z$. Real numbers have $z = z^\star$.

Two "coordinate systems" (x, y), (r, θ) to represent points $z \in \mathbb{C}$:

$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

Here $r = |z| \equiv \sqrt{x^2 + y^2}$ is the modulus or amplitude of z and $\theta = \arg z$ its argument or phase.



Complex numbers

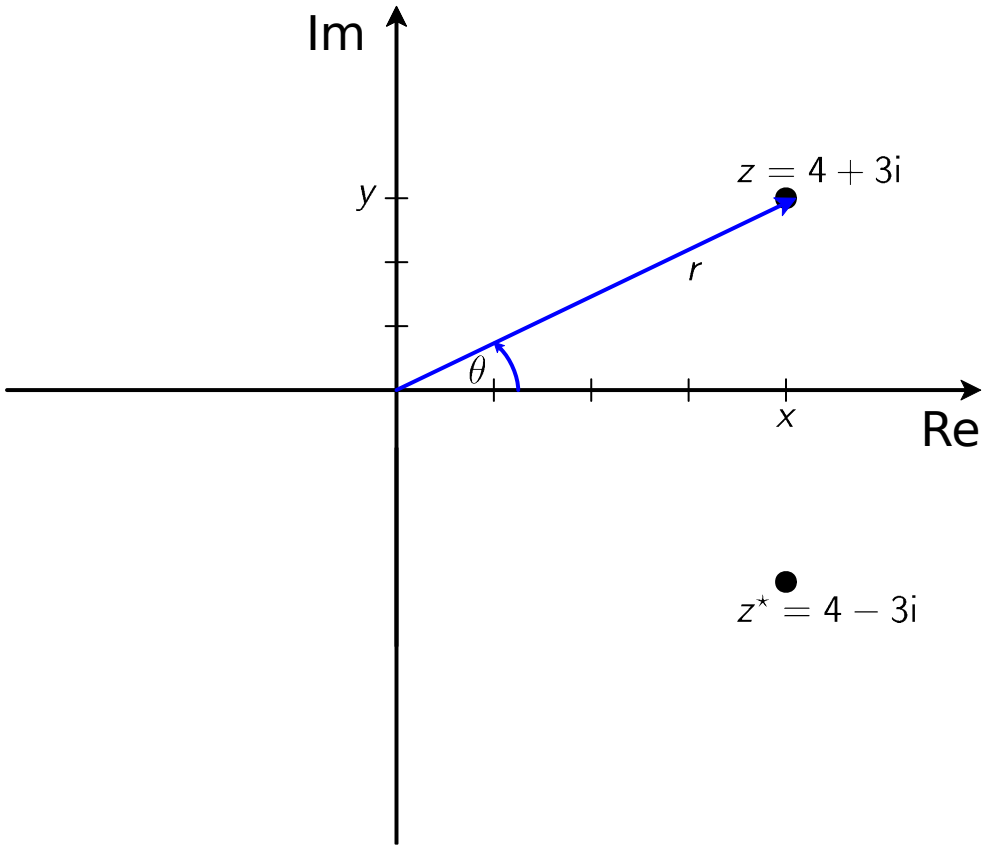
Complex numbers inhabit the complex plane.

Every $z \in \mathbb{C}$ has a complex conjugate $z^\star$ obtained by reflecting z about the Re axis: $\operatorname{Im} z^\star = -\operatorname{Im} z$. Real numbers have $z = z^\star$.

Two "coordinate systems" (x, y), (r, θ) to represent points $z \in \mathbb{C}$:

$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

Here $r = |z| \equiv \sqrt{x^2 + y^2}$ is the modulus or amplitude of z and $\theta = \arg z$ its argument or phase.



Conventions:

- $|z| \geq 0$ — $\sqrt{}$ means +ve square root!
- $-\pi \leq \arg z \leq \pi$.
  - Don't cross -ve real line when measuring θ!

Complex numbers

Some useful relations involving the conjugate:

$$\begin{aligned} \operatorname{Re} z &= \frac{1}{2}(z + z^\star), \\ \operatorname{Im} z &= \frac{1}{2\mathrm{i}}(z - z^\star), \\ |z|^2 &= z z^\star, \\ \arg z^\star &= -\arg z. \end{aligned}$$

To calculate z_1/z_2 multiply top and bottom by $z_2^\star$ giving $\frac{1}{|z_2|^2}(z_1 z_2^\star)$.

Complex numbers

Some useful relations involving the conjugate:

$$\begin{aligned}
\operatorname{Re} z &= \frac{1}{2}(z + z^\star),\\
\operatorname{Im} z &= \frac{1}{2\mathrm{i}}(z - z^\star),\\
|z|^2 &= z z^\star,\\
\arg z^\star &= -\arg z.
\end{aligned}$$

To calculate z_1/z_2 multiply top and bottom by $z_2^\star$ giving $\frac{1}{|z_2|^2}(z_1 z_2^\star)$.

Multiplication and division is easy in polar form:

$$\begin{aligned}
|z_1 z_2| &= |z_1||z_2|\\
\arg(z_1 z_2) &= \arg z_1 + \arg z_2 \pmod{2\pi}.\\
|z_1/z_2| &= |z_1|/|z_2|\\
\arg(z_1/z_2) &= \arg z_1 - \arg z_2 \pmod{2\pi}.
\end{aligned}$$

Exercise: derive de Moivre's formula: $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$.

Back to polynomial equations

Recall that we started with linear equations $ax + b = 0$.

If $a, b \in \mathbb{Z}$ then we don't necessarily have solutions $x \in \mathbb{Z}$.

But expanding the solution space to $x \in \mathbb{Q}$ (or $\mathbb{R}$ or $\mathbb{C}$) allows us to draw coefficients a, b from the same space.

Then we considered quadratic equations $ax^2 + bx + c = 0$.

The solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \in \mathbb{C}$ in general, even when $a, b, c \in \mathbb{C}$.

What about cubic and higher-order equations? Are their solutions some form of yet more "complex" number?

Fundamental theorem of algebra

Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with coefficients $a_k \in \mathbb{C}$.

Fundamental theorem of algebra: if at least one of $a_1, \ldots, a_n \neq 0$ then there is at least one $z_0 \in \mathbb{C}$ for which $f(z_0) = 0$.
Proof: take Short Option S1 in Trinity Term!

Fundamental theorem of algebra

Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with coefficients $a_k \in \mathbb{C}$.

Fundamental theorem of algebra: if at least one of $a_1, \ldots, a_n \neq 0$ then there is at least one $z_0 \in \mathbb{C}$ for which $f(z_0) = 0$.

Consequence: we can factorize $f(z) = c \prod_{k=1}^{n} (z - \lambda_k)$ in which c is a constant and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the (possibly repeated) roots of $f(z)$.

Proof (induction). Let $f_n(z) = f(z)$ and let λ_n be root of this f_n. Then

$$f_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$
$$0 = a_0 + a_1 \lambda_n + a_2 \lambda_n^2 + \cdots + a_n \lambda_n^n,$$

subtract:
$$f_n(z) = a_1 (z - \lambda_n) + a_2 (z^2 - \lambda_n^2) + \cdots + a_n (z^n - \lambda_n^n).$$

Now use $z^k - \lambda_n^k = (z - \lambda_n) q_{k-1}(z)$,
where $q_{k-1}(z) = \sum_{l=0}^{k-1} z^l \lambda_n^{k-1-l}$ is a poly of order $k-1$ in z to write

$$\begin{aligned} f_n(z) &= a_1 (z - \lambda_n) + a_2 (z - \lambda_n) q_1(z) + \cdots + a_n (z - \lambda_n) q_{n-1}(z) \\ &= (z - \lambda_n) \underbrace{\left[a_1 + a_2 q_1(z) + \cdots + a_n q_{n-1}(z) \right]}_{\equiv f_{n-1}(z)}. \end{aligned}$$

Repeat for $f_{n-1}(z)$.

Presummary

The sets $\mathcal{F} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ equipped with $+ - \times \div$ operations have following in common

- Additive and multiplicative identities: $0, 1 \in \mathcal{F}$ with $a + 0 = a$ and $1 \times a = a$ for all $a \in \mathcal{F}$.
- Additive inverse: for any $a \in \mathcal{F}$ there is another $(-a) \in \mathcal{F}$ with $a + (-a) = 0$.
- Multiplicative inverse: for any $0 \neq a \in \mathcal{F}$ there is another $a^{-1} \in \mathcal{F}$ with $a \times a^{-1} = 1$.
- Addition and multiplication are associative and commutative

$$a + (b + c) = (a + b) + c$$
$$a \times (b \times c) = (a \times b) \times c$$
$$a + b = b + a$$
$$a \times b = b \times a.$$

$\times$ is distributive over $+$:

$$a \times (b + c) = a \times b + a \times c.$$

Summary

We've gone from $\mathbb{Z}$ through $\mathbb{Q}$ and $\mathbb{R}$ to $\mathbb{C}$.

You should

- be at ease with basic complex arithmetic: [+ − × ÷ and conjugation *]
- switch effortlessly between $x + \mathrm{i}y$ and $r(\cos\theta + \mathrm{i}\sin\theta)$ forms
- accept that $\sum_{k=0}^{n} a_k x^k = c(x-\lambda_1)\cdots(x-\lambda_n)$ for some (possibly repeated) $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.
  - For example, $x^2 + 2x + 1 = (x+1)(x+1)$ has roots $\lambda_1 = \lambda_2 = -1$.

Read, e.g., RHB Chapter 3 for revision.

Further exploration: countable $(\mathbb{Z}, \mathbb{Q})$ versus uncountable $(\mathbb{R}, \mathbb{C})$ sets.

Sometimes the act of counting modifies the physical system, but this may be reversible.

Complex numbers

john.magorrian@physics.ox.ac.uk

Three lectures:

1. Fundamental ideas (revision)
2. Some important functions of a complex variable
3. Applications

The exponential function

$$\exp : \mathbb{C} \to \mathbb{C}$$

$$\exp z \equiv \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Properties

(1) $\exp(0) = 1$

(2) $\exp(a + b) = \exp(a)\exp(b)$

(3) $\frac{\mathrm{d}}{\mathrm{d}z} \exp z = \exp z$

Because of (2) we can write $\exp z = \mathrm{e}^z$ with $\mathrm{e} = \exp(1)$ and $\mathrm{e}^{a+b} = \mathrm{e}^a\mathrm{e}^b$

Exercise: Directly from defn above prove (2) and (3)!

Trigonometric functions

Setting $z = \mathrm{i}\theta$ in the series expansion gives Euler's formula:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &= \sum_{\text{even } k=0}^{\infty} \frac{(\mathrm{i}\theta)^k}{k!} + \sum_{\text{odd } k=0}^{\infty} \frac{(\mathrm{i}\theta)^k}{k!} \\
&= \sum_{l=0}^{\infty} \frac{(\mathrm{i}\theta)^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{(\mathrm{i}\theta)^{2l+1}}{(2l+1)!} \\
&= \cos\theta + \mathrm{i}\sin\theta,
\end{aligned}$$

using $\mathrm{i}^2 = -1$.

An even more compact notation for the "polar form" of a complex number is $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r = |z|$, $\theta = \arg z$. Notice that $z^\star = r\mathrm{e}^{-\mathrm{i}\theta}$.

Directly from Euler's formula it follows that

$$\cos\theta = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) = \cosh(\mathrm{i}\theta), \qquad \mathrm{i}\sin\theta = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right) = \sinh(\mathrm{i}\theta),$$

and $\cosh z = \cos(\mathrm{i}z)$, $\mathrm{i}\sinh z = \sin(\mathrm{i}z)$.

Logarithm

The function $\log : \mathbb{C} \to \mathbb{C}$ is the inverse of $\exp$: if $a = \exp x$ then $\log a = x$.

Applying $\log$ to both sides of $\exp(x + y) = \exp x \exp y$ gives

$$\log a + \log b = \log(ab) \qquad (*).$$

The easiest way to calculate $\log z$ is to express z in polar form. Let

$$z = re^{i(\theta + 2n\pi)},$$

where $r = |z|$, $\theta = \arg z$, $n \in \mathbb{Z}$ (each n produces the same z). Applying $\log$ to both sides and using $(*)$ we have that

$$\log z = \log |z| + i\left(\arg z + 2\pi n\right).$$

That is, $\log z$ is a multivalued "function"!

Complex powers

For $z, \alpha \in \mathbb{C}$ we define the (complex) power

$$\begin{aligned} z^{\alpha} &\equiv \exp\big(\alpha \log z\big) \\ &= \exp\left[\alpha \log|z| + \mathrm{i}\alpha\left(\arg z + 2n\pi\right)\right], \end{aligned}$$

where $n \in \mathbb{Z}$.

Exercise: How many possible values of z^{α} are there when α is (a) integer, (b) the rational number $\frac{m}{n}$, (c) irrational?

Solving complex equations

Some exercises:

1. Solve $|z + \mathrm{i}|^2 = 4$
2. Sketch the solution to $\arg z = \pi/4$
3. Solve $z^3 = 1 + \mathrm{i}$
4. Calculate $\mathrm{i}^{-\mathrm{i}}$
5. Express $\sin^{-1} z$ as a complex logarithm.

Complex numbers

john.magorrian@physics.ox.ac.uk

Three lectures:

1. Fundamental ideas (revision)
2. Some important functions of a complex variable
3. Applications

Trigonometric identities

You should be able to express $\cos m\theta, \sin m\theta$ as sums of powers of $\cos\theta, \sin\theta$:

- e.g., $\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$.

And powers of $\cos\theta, \sin\theta$ as sums of $\cos m\theta, \sin m\theta$:

- e.g., $\sin^3\theta = -\frac{1}{4}(\sin 3\theta - 3\sin\theta)$.

(Good for simplifying integrals...)

There are less obvious applications. For example, by considering the imaginary part of $S = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\theta}$ show that

$$\sum_{k=1}^{n} \sin k\theta = \frac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

Roots of polynomials

A consequence of the fundamental theorem of algebra is that the polynomial equation,

$$0 = az^n + bz^{n-1} + \cdots + f, \qquad (*)$$

can be written as (assuming $a \neq 0$)

$$\begin{aligned} 0 &= a\prod_{k=1}^{n}(z-\lambda_k) \\ &= az^n - a[\lambda_1 + \cdots + \lambda_n]z^{n-1} + \cdots + a(-1)^n\lambda_1\lambda_2\cdots\lambda_n, \end{aligned}$$

in which the roots $\lambda_1, \ldots \lambda_n \in \mathbb{C}$.

Comparing to the original $(*)$, we see that

$$\sum_{k=1}^{n}\lambda_k = -\frac{b}{a}, \quad \prod_{k=1}^{n}\lambda_k = (-1)^n\frac{f}{a}.$$

That is, the z^{n-1} and z^0 factors tell us about the sum and product of the roots, respectively.

This can have interesting applications.

Polynomials in disguise

Example: by finding the roots of

$$z^{2n+1} + 1 = 0,$$

show that

$$\sum_{k=-n}^{n} \cos \frac{(2k+1)\pi}{(2n+1)} = 0.$$

Applications in physics

Applying a voltage $V(t)$ across a lossless capacitor C produces a current

$$I(t) = C\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

If we set $V = V_0 \mathrm{e}^{\mathrm{i}\omega t}$ then $I = \mathrm{i}\omega CV$: the capacitor obeys a generalised Ohm's law with impedance

$$Z_C = \frac{V}{I} = \frac{1}{\mathrm{i}\omega C}.$$

Applications in physics

Applying a voltage $V(t)$ across a lossless capacitor C produces a current

$$I(t) = C\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

If we set $V = V_0 \mathrm{e}^{\mathrm{i}\omega t}$ then $I = \mathrm{i}\omega CV$: the capacitor obeys a generalised Ohm's law with impedance

$$Z_C = \frac{V}{I} = \frac{1}{\mathrm{i}\omega C}.$$

Similarly, driving current $I(t)$ through a lossless inductor L induces an emf

$$V(t) = L\frac{\mathrm{d}I(t)}{\mathrm{d}t},$$

and so the inductor has impedance $Z_L = \mathrm{i}\omega L$.

Applications in physics

Applying a voltage $V(t)$ across a lossless capacitor C produces a current

$$I(t) = C\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

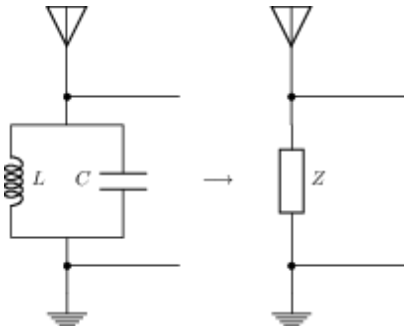
If we set $V = V_0 \mathrm{e}^{\mathrm{i}\omega t}$ then $I = \mathrm{i}\omega CV$: the capacitor obeys a generalised Ohm's law with impedance

$$Z_C = \frac{V}{I} = \frac{1}{\mathrm{i}\omega C}.$$

Similarly, driving current $I(t)$ through a lossless inductor L induces an emf

$$V(t) = L\frac{\mathrm{d}I(t)}{\mathrm{d}t},$$

and so the inductor has impedance $Z_L = \mathrm{i}\omega L$.



$$\begin{aligned}\frac{1}{Z} &= \frac{1}{Z_C} + \frac{1}{Z_L} \\ &= \mathrm{i}\omega C + \frac{1}{\mathrm{i}\omega L} \\ &\to 0, \quad \omega \to 1/\sqrt{LC}.\end{aligned}$$

[ω? Fourier!]

Applications in physics

In quantum mechanics, the state of a system can be expressed as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} a_i(t)|n\rangle,$$

where $|0\rangle$, $|1\rangle$, $|2\rangle$, ... are its energy eigenstates and $a_0(t)$, $a_1(t)$, ... are complex probability amplitudes.

[Aside: $|\psi\rangle$ is actually a *vector* in the complex vector space spanned by the $|n\rangle$.]

Applications in physics

In quantum mechanics, the state of a system can be expressed as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} a_i(t)|n\rangle,$$

where $|0\rangle$, $|1\rangle$, $|2\rangle$, ... are its energy eigenstates and $a_0(t)$, $a_1(t)$, ... are complex probability amplitudes.

[Aside: $|\psi\rangle$ is actually a *vector* in the complex vector space spanned by the $|n\rangle$.]

If we measure the energy of the system, the probability of finding it in state $|n\rangle$ is

$$p_n(t) \propto |a_n(t)|^2.$$

The evolution of the $a_n(t)$ is governed by Schrödinger's equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle.$$

Summary

Complex numbers make beautiful mathematics

- take Short Option S1, *Functions of a Complex Variable,* in TT!

and they're inescapable in modern physics.

Key ideas:

- Re, Im, modulus, arg
- Defn of exp; Euler formula, de Moivre
- Splendid, multivalued logs
- Application to trig
- Fundamental thm of algebra, applications.

Make sure you can do the problems! See also RHB §3.8

Mistakes? Comments? **john.magorrian@physics.ox.ac.uk**