First Year Physics: Prelims CP1

Classical Mechanics: Moritz Riede[1]

Introductory Problems

These problems do not require knowledge beyond A-Level physics and maths. They are intended for revision before or early in MT.

1. A rocket is fired vertically upwards with constant acceleration from the earth's surface. After one minute it reaches a height of 36 km, and the motor is shut off. Neglect air resistance and variation of g with height.

(a) Calculate the maximum height reached.
(b) Calculate the total time of flight.
(c) Draw a $v - t$ diagram for the entire flight.
[Ans: (b) 109.4 km, (c) 5.53 min]

2. Consider the motion of a object able to move along a line under constant acceleration a. At time $t = 0$, its initial position and velocity are x_0 and v_0 respectively. Show that at a later time t,

$$v = v_0 + at, \quad x = x_0 + v_0 t + \frac{1}{2}at^2.$$

Eliminate the time between these two relations to show that

$$a(x - x_0) = \frac{1}{2}(v^2 - v_0^2).$$

Use this to find the minimum stopping distance for a car travelling at $100\,\mathrm{km h^{-1}}$ if its maximum deceleration is $10\,\mathrm{m s^{-2}}$. [Ans 38.6 m]

3. Estimates of inertia and forces:

(a) A cabin cruiser of 15 tonnes (1 tonne = 10^3 kg) drifts in towards a dock at a speed of 0.3 $\mathrm{m\,s^{-1}}$ after its engines have been stopped. A woman on the dock is able to reach the boat with a boathook when it is 2 m from the dock and thereafter can push with a constant force of 400 N. Can she bring the boat to rest before it hits the dock?

(b) A man of mass 80 kg jumps down on a paved road from a height of 0.5 m. He does not bend his knees on landing so his motion is arrested in about 2 cm. What is average force on his bone structure?

(c) If the man now tries jumping from a height of 1.5 m but does bend his knees on impact so that his centre of gravity descends a further distance h after his feet touch the ground, what must h be so that the average force exerted on him by the impact is only five times his normal weight?

[Ans: (a) Yes, (b) 1.96×10^4 N, (c) 0.375 m.]

4. When a car corners at speed one is relying on the friction between the tyres and the road to counter the centrifugal force tending to throw the car outwards. One way to reduce the dependence on friction is to 'bank' the corner. For a car travelling with a speed v entering a corner of radius r, show that the angle of the bank, θ, (measured from the horizontal) should satisfy $\tan\theta = v^2/(gr)$ if there is to be no reliance on friction.
In the light of this result discuss the design of the 'Wall of Death' fairground attraction in which a motorbike is ridden at speed around the inner surface of a large vertical cylinder, with the bike and rider at a small angle to the horizontal plane. What role does friction play and why is a small angle necessary?

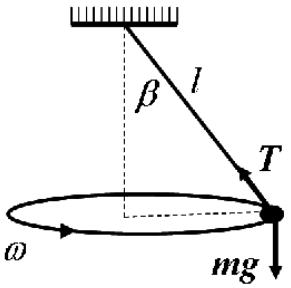
[1] Many thanks to Prof. Neville Harnew who taught the course before and provided his material on which this course is based.

5. Calculate the following centripetal accelerations as fractions or multiples of g and comment on the results: (a) The acceleration towards the Earth's axis of an object resting on the Earth's surface at 45° latitude. The Earth's radius is 6378 km. (b) The acceleration of the Moon towards the Earth. The radius of the Moon's orbit is 384399 km and its period is 27.3 days. (c) The acceleration of an electron moving around a proton at a speed of $0.007c$ where c is the speed of light in an orbit of radius 0.05 nm (the first Bohr orbit of the H atom).

[Ans: (a) 2.4×10^{-3}, (b) 2.8×10^{-4}, (c) 9×10^{21}.]

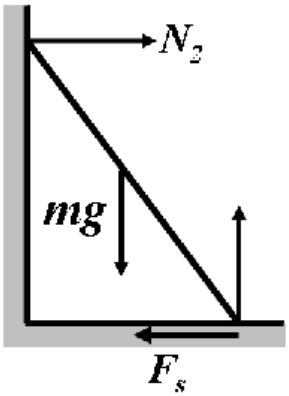
6. An object fixed with respect to the surface of a planet identical in mass and radius to the Earth, experiences zero apparent gravity at the equator. What is the length of a day on that planet? What apparent gravity would be experienced by an object at the planet's poles? [Apparent gravity is the force on a stationary object taking into account both gravitational attraction and rotation.]

7. The muzzle velocity of a gun is 60 $\mathrm{m\,s^{-1}}$. A woman shoots one shot each second straight up into the air, which may be considered frictionless. How many bullets will be in the air at any time and at what height above the ground will they pass each other?



8. A conical pendulum consists of a mass m suspended by a massless string of length l as shown. The mass rotates in a horizontal circle at fixed angular velocity ω so that the string makes a constant angle β with the vertical. Show that the angular velocity of rotation is given by $\omega = \sqrt{g/l\cos\beta}$.

9. A ladder leans against a frictionless wall while the bottom rests on a horizontal floor with coefficient of friction μ as shown. Show that the smallest angle that the ladder can make with the floor without slipping is given by $\tan\theta_{min} = 1/2\mu$.



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Problems I: Introductory problems & collisions in one dimension

Questions 1-9 are standard examples. Questions 10-12 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or may require significant algebra. All topics are covered in lectures 1-5 of Michaelmas term.

Introductory problems

1. Vectors in two dimensions: The position vector of a particle moving in the x-y plane is given by a vector of magnitude r and an angle θ with the x-axis. At $t = 0, \mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$ (cm).

(a) Find a unit vector $\hat{\mathbf{u}}$ that makes an angle $\theta = 20^\circ$ with $\mathbf{r}$.

(b) Show by geometrical construction that any vector in the x-y plane may be obtained by linear combination of $\mathbf{r}$ and $\hat{\mathbf{u}}$.

(c) Assume that at t=0 the particle starts to move in circular motion about the origin, at a *constant* speed $v = 15\ \mathrm{cm\,s^{-1}}$. Explain why the motion is still *accelerated*. Show by geometrical illustration that the velocity is tangent to the path.

(d) Find the dependence of θ on time, hence obtain expressions for $\mathbf{r}$ and the velocity $\mathbf{v}$ as a function of time.
[Ans: (a) $\hat{\mathbf{u}} = 0.55\mathbf{i} + 0.84\mathbf{j}$ (or $\hat{\mathbf{u}} = 0.96\mathbf{i} + 0.29\mathbf{j}$), (d) $\theta = 3t + 0.64$ (rad)]

2. Dimensional analysis

(a) Consider the period of a simple pendulum with length ℓ and bob of mass m moving under gravity. Show on dimensional grounds that a likely relationship for the period is

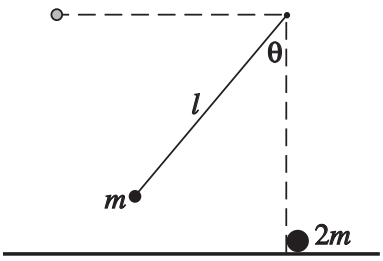
$$T = kf(\theta_0)\sqrt{\ell/g}, \quad \text{where } \theta_0 \text{ is the angular amplitude,}$$

k a dimensionless constant and f an arbitrary function.

(b) For a satellite with mass m, in a circular orbit of radius R about a planet of mass M ($M \gg m$), use the method of dimensions to show that the period of revolution is given by $T^2 = k\dfrac{R^3}{GM}$, where G is Newton's constant.

(c) Use Coulomb's law for the force between two charges q_1, q_2 a distance r apart, $F_E = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r^2}$, to find the dimensions of ϵ_0. Next, use the expression for the force between two parallel wires of length ℓ, a distance d apart, carrying currents I_1, I_2, $F_M = \dfrac{\mu_0}{4\pi}\dfrac{I_1 I_2}{d}\ell$, to find the dimensions of μ_0. Finally, show that the combination $\sqrt{\dfrac{1}{\mu_0\epsilon_0}}$ has dimensions $[\mathrm{LT^{-1}}]$ and find its numerical value from a table of physical constants. Comment on the result.

3. Energy conservation: A pendulum bob of mass m, at the end of a string of length ℓ, starts from rest with the string taut and horizontal – see figure.



At the lowest point of swing the bob strikes a stationary object of mass $2m$ resting on a frictionless horizontal surface. The collision is elastic. Find (using energy considerations):

(a) the speed of the bob just before impact,

(b) the tension in the string at the same instant,

(c) the velocity given to the object by the impact.

(d) After the impact the bob will recoil to reach a new maximum height before resuming oscillations. Find the maximum angle (to the vertical) that the string makes in subsequent oscillations of the bob.

[Ans: (a) $\sqrt{2g\ell}$, (b) $3mg$, (c) $2\sqrt{2g\ell}/3$, (d) 27.3°.]

4. The Simple harmonic oscillator: A particle of mass m constrained to move in the x-direction only is subject to a force $F(x) = -kx$, where k is a constant. Show that the equation of motion can be written in the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \quad \text{where} \quad \omega_0^2 = k/m \; .$$

(a) Show by direct substitution that the expression

$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

where A and B are constants, is a solution and explain the physical significance of the quantity ω_0. Show that an alternative solution may be expressed as

$$x = x_{\max}\cos(\omega_0 t + \phi)$$

where $x_{\max}$ is the amplitude and ϕ is the phase constant of oscillation.

(b) Find in terms of m and ω_0 the change in the potential energy $U(x) - U(0)$ of the particle as it moves from the origin. Explain the physical significance of the sign in your result.

(c) The potential energy is subject to an arbitrary additive constant; it is convenient to take $U(0) = 0$. The kinetic energy $T(x) = \frac{1}{2}mv^2$. Show by differentiation that the particle's total energy ($E = T+V$) is constant. Express E as a function of the particle's (a) maximum displacement $x_{\max}$ and (b) maximum velocity $v_{\max}$.

5. The potential energy function: A particle of mass m, starting from $x = -\infty$, approaches a force region whose potential energy is given by

$$U(x) = \frac{U_0\, a^2}{x^2 + a^2} \; ,$$

where $U_0 > 0$ and a are constants.

(a) Derive an expression for the force on the particle as a function of x.

(b) Draw rough graphs of the force and the potential as functions of x.

(c) State in which region the force is attractive or repulsive, and explain how your answers can be understood in terms of your graphs.

(d) If a particle is released at rest from the origin, displaced from equilibrium, with what velocity will it be travelling at very large distances?

(e) What is the least velocity it must be given at $x = -\infty$ which will allow it to reach $+\infty$?

6. A two-particle problem in 1-D - the centre of mass system: Consider two particles m_1 and m_2 moving in one dimension under the influence of an attractive force of magnitude F_{int} between them. Each is also acted on by an external force: F_1 and F_2, respectively. Let the particles have co-ordinates x_1, x_2 respectively.

(a) Write down the equations of motion of the two masses separately. By adding your two expressions, get a single equation of motion for the system in terms of F_1 and F_2.

(b) Hence show that if there are no external forces present (i.e. $F_1 = F_2 = 0$) the momentum of the system is conserved.

(c) Write down an expression for the position $X_{\rm cm}$ of the centre of mass (CM) of the system. By differentiating your expression with respect to time, show that the total momentum of the system is the same as that of a mass $M = m_1 + m_2$ moving with the velocity of the centre of mass.

(d) Differentiate again, and compare your result with the equation obtained in (a) above. Hence show that the acceleration of the centre of mass is as if all the mass were concentrated there and the resultant external force acted through it. What happens to the centre of mass of the system if there are no external forces?

(e) Assume that the two masses are isolated in an inertial reference frame (no external forces: $F_1 = F_2 = 0$) and consider the motion of the particles in the CM frame using the coordinates $x_i' = x_i - X_{\rm cm}$ $(i = 1, 2)$. Show that the equation of motion of the system in the CM frame under the influence of the internal force is given by Newton's second law as $\mu\ddot{x}' = F_{int}$ where μ is the *reduced mass* of the two particle system, $1/\mu = 1/m_1 + 1/m_2$ and $x' = x_1' - x_2'$ is the relative distance between the two particles.

Collisions in one dimension

7. Elastic equal mass collision: A runaway (assume frictionless) railway truck A of mass m is moving along the track at velocity $u_\circ$ and hits a stationary truck B of identical mass m. By considering the conservation of linear momentum and energy, assuming the collision between the trucks to be *elastic*, determine the velocities, v_A and v_B, of the two trucks after the collision.

Now consider the same collision as viewed by an observer O' located at the *Centre of Mass* (CM) and moving at constant velocity V_{CM} with respect to B.

(a) Calculate V_{CM} and explain why it remains unchanged after the collision.

(b) Show that the total momentum in the CM frame is zero (hence the alternative name *zero momentum frame*).

(c) Determine the initial and final velocities of the two trucks as viewed by O', and illustrate your solution both before and after the collision with simple, clearly labelled sketches. Notice that in the CM frame, the magnitude of the velocity of each truck remains unchanged after the collision.

(d) Discuss the advantages of observing collisions in the CM frame.

8. Elastic unequal mass collision: We have a similar situation as in the previous question but now truck A has mass m_A and initial velocity u_A. Truck B has mass m_B and is initially at rest. The collision is elastic.

(a) Show the subsequent velocities $v_A,\ v_B$ in the laboratory frame are:
$v_A = (m_A - m_B)u_A/(m_A + m_B), \quad v_B = 2m_A u_A/(m_A + m_B)$.

On the basis of these results, find the velocities in the two limiting cases:
(i) $m_A >> m_B$; (ii) $m_A << m_B$.

(b) Find the subsequent initial and final velocities $u_A',\ u_B'$ and $v_A',\ v_B'$ in the CM Frame.

9. Inelastic equal mass collision: Consider the problem of the colliding equal-mass railway trucks in problem 7. Suppose now that the collision is *inelastic*, such that half the initial kinetic energy is 'lost' during the collision.

(a) By considering the conservation of linear momentum and the changes in the total kinetic energy of the system, determine the velocities, v_A and v_B, of the two trucks after the collision in the laboratory frame. Discuss where the 'lost' kinetic energy might have gone (in what sense is it really 'lost'?).

(b) Newton's coefficient of restitution e in a 2-body collision is defined as the ratio of the magnitude of the relative velocity of separation to the magnitude of the relative velocity of initial approach:

$$e = \frac{|\mathbf{v_B} - \mathbf{v_A}|}{|\mathbf{u_B} - \mathbf{u_A}|}.$$

Determine the value of e for the inelastic collision in (a) above.

(c) Consider the same inelastic collision in (a) as viewed by the an observer in the CM frame. What are the final velocities v'_A and v'_B of the two trucks in this frame. Verify that the value of the coefficient of restitution e' in this frame is the same as in part (b).

Additional questions

10. Energy loss to rest: A highly (though not perfectly) elastic ball is released from rest at a height h above the ground and bounces up and down. With each bounce a fraction f of its kinetic energy just before impact is lost. Hence show that the height reached on the nth bounce is $h_n = (1-f)^n h$. Find the time taken for the ball to drop from height h_n and subsequently reach h_{n+1}. Hence find an expression for the time taken for the ball to come to rest and evaluate it for the case when h is 5 m and f is 0.1.

[Ans: time to rest is $\sqrt{2h/g}(1+\epsilon)/((1-\epsilon)$ where $\epsilon = \sqrt{1-f}$; 38.3s.]

11. * Force and momentum - falling chain:

(a) A uniform chain of length L and mass m is stretched out on a frictionless horizontal table with part of its length h hanging down through a hole in the table. Assuming that the chain is released from rest, how long will it take the chain to fall off. Neglect the friction between the hole and the chain.

(b) The chain in (a) is held stretched vertically just above the surface of a weighing scale and then released from rest. What is the reading of the scale when half of the length of the chain has fallen down?

[Ans: (a)$\sqrt{L/g}\cosh^{-1}(L/h)$ (b) $\frac{3}{2}mg$]

12.* The equation of motion: constant force-sliding blocks: A block of mass m slides on the frictionless surface of an inclined plane of angle θ, which itself has a mass M. The plane slides on a horizontal table. Assuming no friction between the surfaces:

(a) Write the equations of motion of the block and the inclined plane in vectorial form as viewed in the inertial reference frame of the table.

(b) By resolving the equations in (a) into components parallel and perpendicular to the surface of the inclined plane, or otherwise, find the acceleration of the block along the plane and also that of the inclined plane along the table.

(c) Calculate the internal force that the block and the inclined plane apply on each other.

(d) Show that the horizontal component of the linear momentum of the total system, ie. the plane plus block, remains constant.

[Ans: (b) $g\frac{\sin\theta(1+\frac{M}{m})}{\sin^2\theta+\frac{M}{m}}$, $-g\frac{\sin\theta\cos\theta}{\sin^2\theta+\frac{M}{m}}$, (c)$Mg\frac{\cos\theta}{\sin^2\theta+\frac{M}{m}}$]

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Problem Set II: Collisions in two dimensions & applications of the equation of motion

Questions 1-10 are standard examples. Questions 11-14 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or may require significant algebra. All topics are covered in lectures 6-9 of Michaelmas term.

Collisions in 2-D

1. Elastic collisions in 2D:
(a) Show that in an elastic collision of two particles masses m_1 and m_2 viewed in the *centre-of-mass frame*, the magnitude of the velocity of each particle is unchanged by the collision.

(b) A particle P_1 mass m with vector velocity $\mathbf{u}_1$ collides elastically with an identical particle P_2 also of mass m that is at rest. The velocities of the particles after the collision are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Working in the *laboratory frame*, show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and hence deduce that, after the collision, either (i) P_1 comes to rest or (ii) P_1 and P_2 move at right angles to each other. Show also that $\mathbf{u}_1 \cdot \mathbf{v}_1 > 0$, and hence deduce that P_1 is scattered through an angle of less than 90° wrt the incident particle.

(c) Solve the problem in (b) by considering the same collision as viewed in the centre-of-mass frame and then transforming to the lab.

2. Collision of an alpha particle with a proton: An alpha particle of mass $4m$ moves at non-relativistic speed and collides elastically and obliquely with a stationary proton, mass m. Draw diagrams illustrating the situation after the collision in the centre-of-mass frame and in the lab frame. Using a geometrical or algebraic method, show that in the lab frame the alpha particle cannot be deflected by more than 14.5° from its original direction of motion.

3. Inelastic collision in 2-D
A particle of mass $2m$ travelling at a non-relativistic speed u in the laboratory collides *inelastically* with a stationary particle of mass m. By considering the collision in the centre-of-mass frame, show that the speed in the laboratory frame of the less massive particle, after the collision, is

$$v_2 = \frac{2u}{3}\left(\cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}\right),$$

where θ is the angle between the particle's trajectory and the original direction of the more massive particle, and α is the coefficient of restitution, defined as the magnitude of the relative velocity of the particles after the collision divided by the magnitude of the relative velocity before the collision.

Work, potential energy and conservation of energy

4. The Work Energy Theorem:

(a) Show that the work done by the force $F(x)$ to move a particle of mass m from a to b is given by: $W_{ab} = T(b) - T(a)$ where $T(x)$ is the kinetic energy at the point x. Does this only apply for conservative forces?

(b) The particle moves towards the origin of the x-axis under the influence of the force $F = k/x^2$ where k is a positive constant. If the velocity at $x = x_\circ$ is u, use the result in (a) to find the distance of minimum approach of the particle to the origin.

5. Gravitational potential of two point masses: A particle P of mass m moves on the x-axis under the combined gravitational attraction of two particle, each of mass M, fixed at the points $(0, \pm a)$ in the $x - y$ plane.

(a) Show that the force acting on the particle P is given by

$$F(x) = -\frac{2mMGx}{(a^2 + x^2)^{3/2}}$$

where G is the gravitation constant.

(b) Find the corresponding potential energy. Discuss the possible motion of the particle using the potential energy curve, showing the regions of bound or unbound motion.

(c) Assuming that P is initially released from rest at $x = 3a/4$, find the maximum speed achieved by the particle in the subsequent motion.

6. SHM about stable equilibrium: The Lennard-Jones potential describes the potential energy between two atoms, each of mass m, in a molecule

$$U(r) = \epsilon[(r_0/r)^{12} - 2(r_0/r)^6],$$

where ϵ and r_0 are constants and r is the distance between the atoms.

(a) Sketch $U(r)$ and show that the position of the minimum potential energy and the depth of the potential well are r_0 and ϵ, respectively.

(b) Expand $U(r)$ as a Taylor series about $r = r_0$ up to the quadratic term.

(c) Use (b) to show that the motion for small displacements about the minimum is simple harmonic and find its frequency. [Ans: $\omega = (12/r_0)\sqrt{\epsilon/m}$.]

(d) Typical vibrational frequencies of diatomic molecules lie in the near infrared (around 3×10^{13} Hz). Estimate the value of the effective spring constant for a typical molecule, such as N_2.
[Ans: $k \sim 400\,\mathrm{Nm}^{-1}$.]

Applications of the equation of motion and resistive forces

7. Projectile in 2D: A ball is thrown with initial speed V, at an angle θ above the horizontal, over flat ground. Show that the motion is restricted to a single plane. Neglecting air resistance, find

(a) the time taken for the ball to reach the ground;

(b) the maximum height reached by the ball;

(c) the horizontal distance travelled by the ball before hitting the ground (range);

(d) the kinetic and potential energies at the position of maximum height (and verify that their sum equals the initial energy).

The following are experimental data on the range and muzzle velocity of mortar shells, all fired at 45° to the horizontal. The time of flight is also tabulated. Compare the ranges and times with the simple treatment above. How would you explain any differences?

Muzzle Velocity (m/s)	Range (m)	Time (s)
101.8	972	14.4
112.2	1160	15.7
121.9	1349	17.0
131.4	1539	18.2

8. A body of mass 1 kg falls from a height of 100 m and reaches the ground with a speed of 40 m s^{-1}. What is the average air resistance? Discuss the meaning of the term 'average' in this connection.

9. An air-filled toy balloon, with a diameter of 30 cm and a mass (not counting the air inside) of about 0.5 g, falls from rest in air.

(a) What is the terminal speed of fall when assuming a linear drag law, taking the balloon to be a sphere and that Stokes' formula for viscous drag is applicable: drag force $F_1 = 6\pi a\eta v$ [N], where a is the radius of the sphere [m], $\eta = 1.7 \times 10^{-5}$ kg m^{-1} s^{-1} is the viscosity of air and v is velocity.

(b) Given this drag law, show that the downward velocity v of the balloon satisfies a differential equation of the form

$$\frac{dv}{dt} = g - \lambda v \ ,$$

and give an expression for the quantity λ. Solve this differential equation, using the appropriate initial condition, and hence find how long it takes for the balloon to come to within 5 per cent of its terminal speed.

(c) How would your result for (a) change if an additional drag force due to turbulent eddies of the form $F_2 = 0.87(av)^2$ [N] (where a is in m and v is in m s^{-1}) were taken into account? Which is more realistic?

10. A ball of mass m is projected vertically upwards at a velocity $v_\circ$. The ball experiences an air resistance force (in addition to gravity) of the form $F_R = -\alpha v^2$ where $\alpha > 0$ is constant and v is the velocity, and reaches a maximum height h before it returns back to the point of projection.

(a) Write down the equations of motion of the ball during its upward and downward journeys and show that the maximum height reached is given by

$$h = a \ln[1 + (v_\circ/v_l)^2]$$

where $v_l = \sqrt{(mg/\alpha)}$ and $a = (v_l)^2/(2g)$.

(b) Show that the velocity of the ball when it returns back to the point of projection is given by

$$v_r^2 = v_l^2[1 - \exp(-h/a)].$$

Additional questions

11.* Gravitational Potential of the Earth Denote the radius of the Earth by R_E and the value of the acceleration due to gravity at the surface by g.

(a) Show that free-fall in the Earth's gravitational field from infinity results in the same velocity at the surface as if the object had dropped a distance R_E under a constant acceleration of $g = GM_E/R_E^2$, where G is the gravitational constant and M_E is the mass of Earth.

(b) Show that for heights h small compared to R_E the first correction to the velocity at the surface for an object in free-fall is: $v = \sqrt{2gh}\left(1 - \dfrac{h}{2R_E}\right)$.

(c) For a height h small compared to R_E show that the fractional change in period of a pendulum clock is given by: $\Delta T/T = h/R_E$. Does the clock gain or lose and evaluate the change for a height of 100 m in seconds per week?

12.* A canon is placed on an inclined plane of angle α with the horizontal. The gun is aligned with the steepest line of the plane and can fire shells in both the upward and downward directions, at an initial speed V and an angle θ with respect to the plane.

(a) Find the distance where the shell will land up the plane for $\theta = \alpha = 30^\circ$ (Ans: $\frac{2}{3}(V^2/g)$)

(b) Show that the ratio of the *maximum* ranges up and down the plane is given by:

$$\frac{R_{up}}{R_{down}} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$$

13. A body of mass M, travelling in a straight horizontal line, is supplied with constant power P and is subjected to a resistance Mkv^2, where v is its speed and k is a constant.

(a) Show that the speed of the body cannot exceed a certain value v_m and find an expression for v_m.

(b) Show that, starting from rest, the body acquires half the maximum speed after travelling a distance $\ln(8/7)/(3k)$.

(c) If the power is then cut off and an additional retarding force of constant value F imposed, find the subsequent time which elapses before the body comes to rest.

[Ans: (a) $v_m^3 = P/(Mk)$; (c) time$= \sqrt{M/Fk}\tan^{-1}[0.5v_m\sqrt{Mk/F}]$.]

14.* Work against non-conservative forces

This question follows on from Question 10:

(a) Calculate, by direct integration over the resistive term, the energy dissipation during the upward journey and show that your result is consistent with the work-energy theorem.

(b) Using the result of Q10 (b) for the final velocity, show that for $v_\circ = v_l$ half of the initial energy is lost during the round trip.

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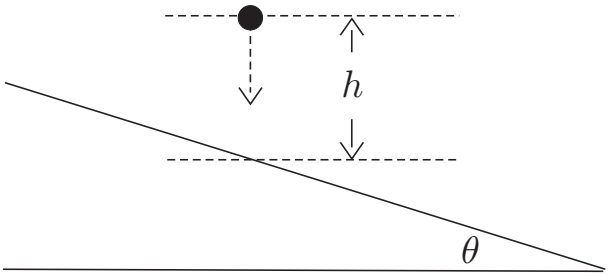
Classical Mechanics: Prof. Armin Reichold

Problem Set III : Projectiles, rocket motion and motion in E & B fields

Questions 1-10 are "standard" examples. Questions 11-12 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require significant algebra. All topics are covered in lectures 1-5 of Hilary term.

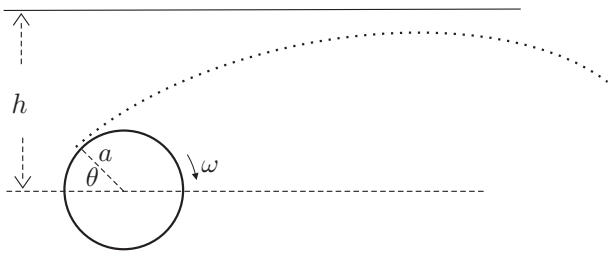
More on projectiles

1. Projectile with no air resistance: A small sphere is released from rest and, after falling a vertical distance h freely under gravity, bounces on a smooth inclined plane (which is at an angle $\theta < 90^\circ$ to the horizontal). Given that the sphere loses no energy on impact, in what direction (relative to the upward normal to the plane) will it move immediately after impact? Show that the distance down the plane between this impact and the next is $8h \sin\theta$.
[Hint: you may wish to choose axes parallel to and perpendicular to the plane.]



2. Projectiles from a rotating wheel: Drops of water are thrown tangentially off the rim of a wheel of radius a, rotating in a vertical plane about its axis, which is fixed and horizontal, with angular velocity $\omega > \sqrt{g/a}$ (as in the diagram). Neglecting air resistance, determine the point on the wheel rim from which drops will rise to the maximum height above the wheel axis (consider the vertical contributions to PE and KE as a function of angle θ). Hence show that if a horizontal ceiling, at height $h > a$ above the axis of the wheel, is not to be spattered with water, ω must satisfy

$$(a\omega)^2 \;<\; gh + g\sqrt{h^2 - a^2}.$$



Motion of Rockets

3. Two masses m and δm ($\delta m \ll m$) are joined together, and moving with velocity v in some inertial frame. At some instant the mass δm flies apart from m (e.g., because of the release of a spring) along the direction of motion with a relative velocity w with respect to the mass m, such that the velocity of m becomes $v + \delta v$. Show that, if the small mass is projected in the backwards direction,

$$m\delta v = w\delta m \ .$$

Explain whether the same expression applies for the case in which the small mass is projected in the forwards direction.

We now turn to the case of a rocket which burns fuel, so that the mass of the rocket remaining at time t is $m(t)$. Show that the acceleration of the rocket is given by the equation

$$m\frac{dv}{dt} + w\frac{dm}{dt} = 0 \ ,$$

paying careful attention to signs. Finally, modify the rocket equation to take account of motion against a constant gravitational field (acceleration due to gravity $= -g$).

4. A rocket of initial mass M, of which half is fuel, is launched vertically upwards at $t = 0$. It burns the fuel at a mass rate of α (a positive quantity) and ejects it backwards from the rocket at a velocity V with respect to the rocket. (a) Show that the equation governing the (vertical) speed v of the rocket is

$$\frac{dv}{dt} = -g + \frac{\alpha V}{M - \alpha t}$$

where g is the acceleration due to gravity, assumed constant. (b) Show that the rocket cannot leave the launch-pad on ignition unless $\alpha V > Mg$. (c) Given $\alpha V > Mg$, show that fuel burn-out occurs at $t = M/2\alpha$. (d) Show that the upward velocity at burn-out is $v = V \ln 2 - gM/(2\alpha)$ (e) Find the height of the rocket at burn-out and the maximum height reached.

[You need $\int \ln x \, dx = x \ln x - x$.]

[Ans for (e) max. height is $(1 - 2\ln 2)(MV/2\alpha) + (V \ln 2)^2/(2g)$]

5.* Two stage rocket: A rocket of total initial mass (casing + fuel + payload) is Nm, where m is the payload mass, and it ejects mass at a constant rate with velocity u relative to the rocket. The initial ratio of (casing mass) to (casing mass + fuel mass) is r. Ignore gravity and air-resistance and assume that the rocket starts from rest.

(i) One stage rocket. Show that the payload achieves a final velocity of:

$$v_1 \ = \ u \ln\Big[\frac{N}{rN + (1-r)}\Big].$$

(ii) Two stage rocket. The total initial mass of both stages (fuel+casing+payload) is now Nm and the total initial mass of the second stage (fuel+casing+payload) is nm. As for the first stage, the second stage ejects mass at a constant rate with velocity u relative to the rocket. Likewise, the initial ratio of (casing mass) to (casing mass + fuel mass) of the second stage is also r. Assuming that the first stage is dropped when its fuel is exhausted show that the final velocity of the payload is:

$$v_2 \ = \ u \ln\Big[\frac{N}{rN + n(1-r)}\Big] \ + \ u \ln\Big[\frac{n}{rn + (1-r)}\Big].$$

(iii) If N and r are constant, show that v_2 is a maximum for $n^2 = N$ and then:

$$v_2^{max} \ = \ 2u \ln\Big[\frac{n}{rn + (1-r)}\Big].$$

(iv) Realistic maximum values of u from chemical combustion are of the order of $2.9\,\mathrm{kms}^{-1}$. If $r = 0.1$ could Earth escape speed be reached with either a one or two stage rocket?

Work and energy

6. Calculation of work along a path: A particle moves subject to a force field $\mathbf{F} = \alpha(2xy, x^2, 0)$, where α is a constant.
The particle moves from the origin O to the point P with coordinates $(1, 1, 0)$. What is the work done by $\mathbf{F}$ on the particle for each of the following paths from O to P: (a) a straight line from O to $(1, 0, 0)$ followed by a straight line to P; (b) a straight line from O to $(0, 1, 0)$ followed by a straight line to P; (c) a straight line direct from O to P.
Repeat for the force field $\mathbf{G} = \alpha(x^2, 2xy, 0)$. Comment on whether the force fields $\mathbf{F}$ and $\mathbf{G}$ are either conservative or non-conservative.

7.* Pendulum Oscillations with large amplitude: Consider the period of a simple pendulum with length ℓ and bob of mass m moving under gravity, but not confined to small–angle oscillations.

It was shown in Problem Set I that, on dimensional grounds, the period of oscillations is given by

$$T = kf(\theta_0)\sqrt{\ell/g}, \quad \text{where } \theta_0 \text{ is the angular amplitude,}$$

k a dimensionless constant and f an arbitrary function.

a) Re-work this problem by using energy conservation or otherwise to show that for small angle oscillations $kf(\theta_0) = 2\pi$. Show that for large oscillations $k = 2\sqrt{2}$ and

$$f(\theta_0) = \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \qquad (\dagger)$$

b) Use the substitutions: $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi$ and $\alpha = \sin\frac{\theta_0}{2}$ to show that the period of oscillation is given by:

$$T = 4\sqrt{l/g}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2\sin^2\phi}}$$

c) Expand the integrand in the above expression to estimate the period of oscillation to second order in θ_0. Estimate the percentage error if using the small angle approximation for $\theta_0 = \pi/3$.

Motion of particles in E & B fields

8. A charged particle in magnetic field: The motion of a particle of mass m and charge q moving in a magnetic field $\mathbf{B} = (0, 0, B)$ is governed by Newton's Second Law in the form

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}\ , \qquad (\dagger\dagger)$$

where $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position of the particle at time t. At $t = 0$ the particle is at the origin and has velocity $(0, v_0, w_0)$.
Write down the three components of equation $(\dagger\dagger)$, and integrate them with respect to time, paying attention to the initial conditions. Show that

$$\ddot{y} + \omega^2 y = 0 \quad \text{where} \quad \omega = qB/m\ ,$$

and hence that $y = (v_0/\omega)\sin\omega t$. Find also x and z, and describe and sketch the path of the particle. Give a simple physical description of the balance of forces acting on the particle.

9. Motion in both E & B fields: An electron moves in a force field due to a uniform electric field $\mathbf{E}$ and a uniform magnetic field $\mathbf{B}$ which is perpendicular to $\mathbf{E}$. Let $\mathbf{E} = (0, E_y, 0)$ and $\mathbf{B} = (0, 0, B_z)$. Find the resulting path $\mathbf{r}(t) = (x(t), y(t), z(t))$ of the electron from the initial conditions $\mathbf{r}(0) = (0, 0, 0)$, $\dot{\mathbf{r}}(0) = (v_0, 0, 0)$.

Proceed as follows: the vector force on an electron, of charge $-e$, in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is $(-e)(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$; this equals $m\ddot{\mathbf{r}}$. Write out the three components of the resulting equation. Show first that $z(t) = 0$ for all t. Next integrate with respect to time, and use the initial conditions, to obtain the following coupled differential equations for $x(t)$ and $y(t)$:

$$m\dot{x} = -eyB_z + mv_0 \ , \qquad m\dot{y} = -eE_y t + exB_z \ . \tag{$*$}$$

Eliminate $\dot{x}$ to get

$$\ddot{y} + \omega^2 y = A \ ,$$

where ω and A are constants that you should determine. Solve this differential equation to obtain $y = a(1 - \cos \omega t)$, where $a = A\omega^{-2}$; substitute in the second of $(*)$ to obtain $x = a \sin \omega t + bt$, where $b = E_y/B_z$.
This path is a cycloid: sketch it in the xy-plane, indicating the directions of $\mathbf{E}$ and $\mathbf{B}$ on your sketch.

Non-inertial reference frames

10. A passenger holding a parcel of mass m is standing in a lift which is being accelerated upward by a constant force F. The total mass of lift plus passenger is M. (Assume $M >> m$.)

(a) What is the acceleration of the lift?

(b) What weight does the parcel appear to have for the passenger?

(c) If the passenger drops the parcel from height h, how long does it take to reach the floor? Try this both in the lift frame (non-inertial) and an inertial frame (eg w.r.t. lift shaft).

(d) If the lift changes to uniform velocity, what must the value of F become? Comment on your result.

Additional questions

11. Curvlinear motion: A particle P is moving along an arbitrary path C with velocity $\mathbf{v(t)}$ and position vector $\mathbf{r(t)}$ with respect to an origin O.

(a) Show that if the particle velocity satisfies the relation $\dot{\mathbf{r}} = \mathbf{c} \times \mathbf{r}$ where $\mathbf{c}$ is a constant vector then P moves in uniform circular motion.

(b) Let $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{n}}$ be respectively the instantaneous tangential and normal unit vectors at a point along the path, and s is the (scalar) length along the path. Show that the derivative of one unit vector with respect to time or space (s) is proportional the other. Hence show that $\dfrac{d\hat{\boldsymbol{\theta}}}{ds} = \dfrac{1}{\rho}\hat{\mathbf{n}}$ where ρ is the radius of curvature at the point.

(c) Use the relation $\mathbf{v} = v\hat{\boldsymbol{\theta}}$ to show that the acceleration of the particle is given by $\mathbf{a} = \dot{v}\hat{\boldsymbol{\theta}} + \dfrac{v^2}{\rho}\hat{\mathbf{n}}$

12.* Projectile with resistance: For a projectile with a linear air resistance force $F_s = -mk\mathbf{v}$, show that the maximum horizontal range is given by the equation

$$\left(v_0 + \frac{g}{k}\right)\frac{x}{u_0} + \frac{g}{k^2}\ln\left[1 - \frac{kx}{u_0}\right] = 0,$$

where $u_0 = V\cos\theta$, $v_0 = V\sin\theta$ are the horizontal and vertical components of the initial velocity. The above equation cannot be solved in closed form, either it is solved numerically or it may be approximated, assuming that the correction with $k \neq 0$ is small. For the latter method show that

$$x_{max} \approx \frac{2v_0u_0}{g} - \frac{8v_0^2u_0}{3g^2}k.$$

First Year Physics: Prelims CP1

Classical Mechanics: Prof. Armin Reichold

Problem Set IV Central Forces and Orbits

Questions 1-11 are "standard" examples. Questions 12-15 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in lectures 6-10 of Hilary term.

1. General circular motion: A particle of mass m is constrained to slide on the inside of a vertical smooth semi- circular ring of radius r. The position of the particle is described by a polar coordinate system whose origin is at the centre of the circle with axes along the orthogonal unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ where θ is the angle between the radius vector $\boldsymbol{r}$ and the the vertical line that passes through the origin.
(a) Derive the general relationships for velocity and acceleration in polar coordinates. Hence show that the acceleration of the particle may be written as

$$\boldsymbol{a} = r\ddot{\theta}\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\hat{\boldsymbol{r}}$$

where $v = r\dot{\theta}$ is the magnitude of the particle velocity.
(b) Use the equation of motion in the $\hat{\boldsymbol{\theta}}$ direction to calculate the period of small oscillation about the bottom point.
(c) Use the equation of motion in the $\hat{\mathbf{r}}$ direction to get an expression for the reaction force exerted by the surface as a function of θ. Assuming the particle is released from rest at the top of the semi-circle, use this expression and the conservation of energy to show that the magnitude of the reaction force at $\theta = 60°$ is $\frac{3}{2}mg$.

2. Motion under a central force: A particle is moving under an attractive force $f(r)$ per unit mass directed along the line joining the particle to a fixed point O. The vector displacement of the particle from O at time t is $\mathbf{r}$. Show that $\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector and hence that the particle moves in a plane. Using plane polar coordinates (r, θ) for motion within the plane: define the angular momentum J; using the transverse equation of motion show that $|J|$ is a constant; show that the radius vector r sweeps out area at a constant rate.

3. Motion on 2-D surface: Two particles of mass m are connected by a light inextensible string of length l. One of the particles moves on a smooth horizontal table in which there is a small hole. The string passes through the hole so that the second particle hangs vertically below the hole.
(a) Use the conservation of energy and angular momentum or otherwise to show that:

$$\dot{r}^2 \;=\; A - B/r^2 - gr$$

where $r(t)$ is the distance of the first particle from the hole, A and B are constants and g is the acceleration due to gravity. Initially the particle on the table is at a distance $l/2$ from the hole and moves with a speed v_0 directed perpendicular to the string. Find the values of A and B.
(b) Show that the condition for the particle on the table to move in circular motion is given by $v_0^2/gl = \frac{1}{2}$.
[Ans: $A = (gl + v_0^2)/2$, $B = (lv_0)^2/8$.]

4. Simple orbits

(a) If a communication satellite is to remain in orbit constantly above a particular city on the equator, what distance above this city will the orbit be? Why must the city be on the equator?

(b) A binary star consists of two stars bound together by gravity. From spectroscopic studies of Plaskett's binary it is known that the period of revolution of the stars about their centre of mass is 14.4 days, the speed of each component is $220\,\mathrm{km\,s^{-1}}$ and they are moving roughly in opposite directions along a nearly circular orbit.

Argue that the masses of the two component stars must be roughly equal. Assuming a circular orbit, find the distance between the two stars and their masses.

[Ans: (a) 35886 km; (b) 8.7×10^{10} m, 1.25×10^{32} kg,]

5. Rotation of galaxies: A galaxy may be modelled as a dilute spherical distribution of stars with constant density within a radius R and total mass M. Find expressions for the velocity v of an individual star of mass m in a circular orbit about the centre of the galaxy at distances r both greater than and less than R. [For the latter calculation you will need to use the equivalent of Gauss Law: the gravitational force at an interior point r is given by the mass contained within a sphere of radius r.] Sketch the dependence of v on r. Observations of $v(r)$ show that for many galaxies the measured curve requires a larger mass than that determined from the luminous matter. This is the so-called *dark matter* problem and it is a hot topic in current astrophysical research.

6. Putting satellite into orbit: It is required to put a satellite into an orbit with apogee (furthest distance from the centre of the planet) of $5R/2$, where R is the radius of the planet. The satellite is to be launched from the surface with a speed v_0 at 30° to the local vertical. If M is the mass of the planet, use the conservation of energy and angular momentum to show that $v_0^2 = 5GM/4R$. Assume that the planet is not rotating and that effects due to the planetary atmosphere can be ignored. (Adapted from French & Ebison).

7. Deflection by a central force

An incoming comet of mass m is detected at the edge of the solar system. It has a velocity v and an impact parameter b (the minimum distance between the centre of the sun and the extrapolation of the comet's initial path) with respect to the Sun (mass M).

(a) Sketch the expected trajectory of the comet. Indicate the impact parameter b and the angle of scattering (deflection) ϕ on your plot.

(b) Show that the total energy of the comet as a function of position vector $\mathbf{r}$ from the centre of the Sun may be written as:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}\,.$$

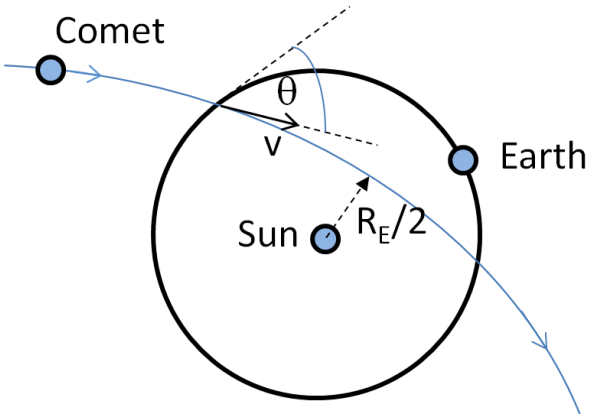
Find expressions for J and α.

(c) Using the result in (b) or otherwise, show that the distance of closest approach to the centre of the Sun is

$$r_{min} = \frac{GM}{v^2}\left[\sqrt{1 + \left(\frac{bv^2}{GM}\right)^2} - 1\right]$$

where G is the gravitational constant.

8. The comet of Question 7 is deflected by the Sun and on its trajectory crosses the Earth's orbit. The distance of closest approach to the Sun is measured to be half the distance between the Earth and the Sun, R_E, assuming that the Earth's orbit is circular. The velocity at that point is 60 kms^{-1}, which happens to be twice the orbital speed of the earth. What is the comet's speed v when it crosses the Earth's orbit? What angle θ does the comets trajectory make with the Earth's orbit at that point? [You may neglect the gravitational attraction between the comet and the Earth.]



9. Changing orbit: A satellite of mass m is put into a circular orbit of radius r_0 around the Earth (of mass M).

(a) Given that the speed of the satellite is v_0, show that

$$r_0 = \frac{GM}{v_0^2},$$

where G is the gravitational constant.

(b) Given that J_0 is the satellite's angular momentum and E_0 its total energy (kinetic plus potential), show that

$$J_0 = \frac{GMm}{v_0}\ , \quad \text{and} \qquad \mathrm{E}_0 = -\frac{1}{2}\mathrm{mv}_0^2$$

(c) Whilst in its circular orbit, the satellite is given an instantaneous impulse which changes its angular momentum from J_0 to αJ_0, where $\alpha < 1$. The satellite's energy remains unchanged. Describe the shape of the new orbit and, using the result of Question 7(b), calculate the minimum and maximum values for the distance between the satellite and the centre of the Earth if $\alpha = 12/13$.

(d) Assuming that whilst the satellite in its circular orbit the impulse leaves the angular momentum unchanged but increases its kinetic energy by 1%. Find the difference between the maximum and minimum distances from the centre of the earth.

Sketch the orbits of (c) and (d) comparing with the original circular orbit.

[Ans: (c) $r_{max}/r_0 = 1 + (1-\alpha^2)^{1/2}$, $r_{min}/r_0 = 1 - (1-\alpha^2)^{1/2}$ (d) $0.20 r_0$]

10. Rutherford scattering: A particle of electric charge q_1 and mass m is projected, with velocity v_0 and impact parameter d ($d > 0$), towards a fixed nucleus of opposite charge q_2 (attractive force). Sketch the trajectory. Show that the angle of deflection α is given by:

$$\cot\left(\frac{\alpha}{2}\right) \ = \ \frac{4\pi\epsilon_0}{q_1 q_2} d v_0^2 m\ .$$

By making the replacement $\dfrac{q_1 q_2}{4\pi\epsilon_0} \to GMm$, notice that the above result may be adapted to find the angle through which the path of a comet of mass m is ultimately deflected as it passes the Sun.

11. Two-body dynamics

Consider an isolated two-body system of masses m_1 and m_2 moving under their internal gravitational interaction. The particles can be treated as point masses and their position in an inertial frame is described by the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$.

(a) Show that the equation of motion for either particle may be written as

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = \mu\ddot{\mathbf{r}}$$

where $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ is the relative position vector and $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

(b) Show that if the motion is viewed in the centre of mass frame, the kinetic energy of the two particles is given by $T = (1/2)\mu\dot{\mathbf{r}}^2$.

(c) Show that if the motion is viewed in the centre of mass frame, the total angular momentum of the system is given by $\mathbf{J} = \mu\mathbf{r} \times \dot{\mathbf{r}}$.

Conclude therefore that the motion of the two particles may be analysed using an equivalent system comprising a *single particle* of mass μ moving under the influence of an attractive central force $f(r)\hat{\mathbf{r}}$ where $\mathbf{r}$ is the position vector of the particle with respect to the other.

Additional questions

12. Elliptical orbit with a spring-like force

A particle of mass m moves in the $(x.y)$ plane under the influence of a spring-like force of the form $\mathbf{F} = -k\mathbf{r}$, where $k > 0$ is a positive constant. Assuming that at $t = 0$ the position of the particle is given by $\mathbf{r}_0 = (x_0, 0)$ and the velocity by $\mathbf{v}_0 = (0, u_0)$,

(a) Write the EOM in Cartesian coordinates and show that the path of motion is an ellipse centred at the origin of coordinates. Find the major and the minor axes of the ellipse.

(b) Derive an expression for the effective potential and use it to analyse the motion and to show that it is always bound.

13.* Motion of a binary star

The two individual stars in a binary system ($m_1 = m_0, m_2 = 2m_0$) are in circular orbit about their common centre of mass and are separated by a distance r_0. At some stage, the more massive star explodes and as a result suffers a spherically symmetric loss of mass. After the explosion the masses of the starts become equal:

(a) Using circular motion dynamics, calculate the total energy and period of the binary star as it is viewed before the explosion in the centre of mass frame.

(b) Now calculate the total energy and period of the equivalent single-particle orbit system before the explosion.

(c) Calculate the total energy of the binary star after the explosion in its new centre of mass frome. Hence show that the binary star will remain intact (bound) after the explosion.

(d) Sketch the equivalent single particle orbit and the actual binary orbits as viewed in the new centre of mass frame after the explosion.

[Ans: (a) $E = -\frac{Gm_0^2}{r_0}, T = 2\pi\sqrt{\frac{r_0^3}{3Gm_0}}$: (b) Same result as (a) : (c) $E = -\frac{1}{4}\frac{Gm_0^2}{r_0}$]

14. An efficient way to reach the Moon is to first put the spacecraft in a low circular Earth orbit (radius r_0, speed v_0). The speed is then increased to v_p via a boost of the rocket, giving an elliptical orbit with apogee r_a at the Moon's orbital radius about the Earth, and perigee at r_0. Show that

$$\left(\frac{v_p}{v_0}\right)^2 = \frac{2r_a}{r_0 + r_a}.$$

Taking $r_0 \approx R_e$, $r_a \approx 60R_e$, calculate v_0 and v_p.

Calculate the fractional error on the spacecraft's apogee with respect to r_0 if v_p/v_0 is incorrect by a fraction f. Hence show that a 0.1% error in the boosted velocity gives a greater than 10% error in the spacecraft's apogee at the Moon.

What other considerations are necessary for a successful rendezvous between spacecraft and Moon? [$R_e = 6.38 \times 10^6$ m] (Adapted from Fowles & Cassidy.)

15. Consider a planet orbiting the sun in an elliptic orbit. Let r and r' be the distance from the planet to the two foci. An alternative and equivalent definition to the standard cartesian equation of an elipse is that $r + r' =$ constant $= 2a$, where a is the length of the major axis. The separation between the two foci of the ellipse is $2a\epsilon$ where ϵ is the eccentricity. Let the angle between the line joining the two foci and a line from one focus to the planet be θ.

(a) Show that

$$\frac{1}{r} = \frac{1 - \epsilon\cos\theta}{a(1-\epsilon^2)}$$

(b) Show that $r^2\dot{\theta} = C$ where C is a constant. Hence using the result in part (a) show that

$$\ddot{r} = -\frac{\epsilon\, C}{a(1-\epsilon^2)}\cos\theta\,\dot{\theta}$$

(c) Using the result in (b) show that the acceleration in the radial direction

$$a_r = -\frac{k}{r^2}$$

and find the constant k. Discuss the signifiance of this result for the law of gravity.

First Year Physics: Prelims CP1

Classical Mechanics: Prof. Armin Reichold

Problem Set V Angular momentum & rotational dynamics

Questions 1-9 are "standard" examples. Questions 10-11 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in lectures 10-15 of Hilary term.

1. A turntable consists of a flat disk A of radius a and mass m. Calculate its moment of inertia about its centre. It is spinning freely with angular velocity ω. Give expressions for its angular momentum and its kinetic energy. A second identical disk B is held, horizontal and stationary, directly above A. B is dropped onto A and sticks to it. Find expressions for the angular velocity, the angular momentum and the kinetic energy of the assembly.

2. A cylinder of mass M, radius r, is set to rotate with angular velocity ω_0 about its own axis, which is fixed. If after time t the angular velocity is ω_1, find the frictional torque N on the cylinder at time t, assuming that:

(a) N is constant,

(b) N is proportional to ω, the instantaneous angular velocity.

[Ans: (a) $Mr^2(\omega_0 - \omega_1)/2t$, (b) $[Mr^2\omega/2t]\ln(\omega_0/\omega_1)$]

3. A cylindrical reel of thread of radius r and mass m is allowed to unwind under gravity, the upper end of the thread being fixed. Find (a) the initial acceleration of the reel (b) the initial tension in the thread. [The moment of inertia of a uniform cylinder of radius r and mass m about its axis is $\frac{1}{2}mr^2$.]
[Ans: (a) $2g/3$, (b) $mg/3$]

4. A space station is located in a gravity-free region of space. It consists of a large diameter, hollow thin-walled cylinder which is rotating freely about its axis. It is spinning at a speed such that the apparent gravity on the inner surface is the same as that on earth. The cylinder is of radius r and mass M.

(a) What is the minimum total work which had to be done to get the cylinder spinning up to speed.

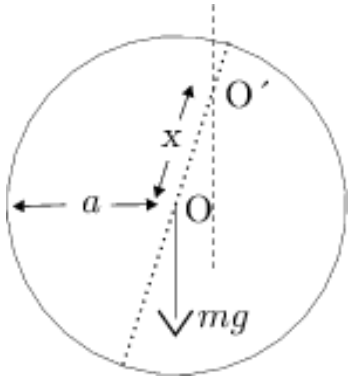
(b) Radial spokes, of negligible mass, connect the cylinder to the centre of rotation. An astronaut, of mass m, climbs a spoke from the inner surface of the cylinder to the centre. What will be the fractional change in the apparent gravity on the surface of the cylinder?

(c) If the astronaut now climbs halfway up a spoke and lets go, how far along the cylinder circumference from the base of the spoke will the astronaut hit the cylinder?
Assume throughout that the astronaut is point-like. [Ans: $(\sqrt{3} - \pi/3)r$]

5. A disk of radius a and mass m is suspended at its centre by a vertical torsion wire which exerts a couple $-c\theta$ on the disk when it is twisted through an angle θ from its equilibrium position. Show that oscillations of the disk are simple harmonic, and obtain an expression for the period.

A wire ring of mass m and radius $a/2$ is dropped concentrically onto the disk and sticks to it. Calculate the changes to (a) the period (b) the amplitude (c) the energy of the oscillations for the two cases where the ring is dropped on (i) at the end of the swing when the disk is instantaneously at rest (ii) at the midpoint of the swing when the disk is moving with its maximum angular velocity.

6. A compound pendulum consists of a uniform circular disk of radius a and mass m with a series of small holes drilled at regular intervals along a diameter. A horizontal axis about which the disk is free to turn is placed through the hole O', a distance x from the centre of the disk (O).



(a) Find the period T of small-angle oscillations using the torque-angular momentum equation,
(b) Sketch the behaviour of T^2 as a function of x for $0 < x < a$. For what values of x is T a minimum? Outline how you would use the measurement of $T(x)$ to measure g.
[Ans: Minimum value of T is at $x = a/\sqrt{2}$.]

7. State and prove the theorem of parallel axes. Find by integration the moment of inertia of a uniform rod of mass m, length b about (i) one end (ii) the centre of gravity. Show that your results are consistent with the theorem.
The rod (one end A, the other B) lies at rest on a smooth horizontal table. A horizontal force F is applied to its centre, perpendicular to its length. Find the initial acceleration of A. The experiment is repeated with the point of application of the force now at B (but still perpendicular to its length). Find the initial acceleration of A. In which direction does it start to move? Find the distance from A at which the force must be applied for the initial acceleration of A to be zero.
[Ans: $mb^2/3$; $mb^2/12$; F/m; $-2F/m$; $2b/3$.]

8. Show that the moment of inertia of a spherical shell of radius R and mass M about an axis through its centre is $\frac{2}{3}MR^2$. Show also that the moment of inertia of a uniform solid sphere of radius R and mass M is $\frac{2}{5}MR^2$. The spheres are allowed to roll (from rest), without slipping a distance L down a plane inclined at a angle θ to the horizontal. Find expressions for the speeds of the spheres at the bottom of the incline and show that $\dfrac{\Delta v}{\langle v \rangle} = 8.7\%$ where Δv is the difference in the speeds and $\langle v \rangle$ is the mean of the two speeds. Which sphere has the larger speed?

9. An interacting system is composed of two bodies of masses m_1, m_2 at positions $\mathbf{r}_1$, $\mathbf{r}_2$ with respect to an origin at O. Show that the angular momentum of the system about an axis through O may be written as

$$\mathbf{L} = (m_1 + m_2)\mathbf{r}_{cm} \times \dot{\mathbf{r}}_{cm} + \mu \mathbf{r} \times \dot{\mathbf{r}},$$

where $\mathbf{r}_{cm}$ is the position of the centre-of-mass of the system (wrt O), $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative coordinate and μ is the reduced mass $1/\mu = 1/m_1 + 1/m_2$.

Additional questions

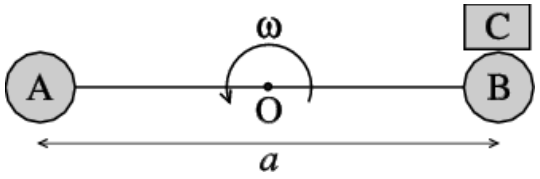
10. A thin uniform bar of length l and mass M is suspended horizontally at rest. It is suddenly released and immediately it is struck by a sharp blow vertically upwards at one end (the duration of the impulse is negligible). The impulse causes the bar to rotate at an angular velocity ω_0 about the centre of mass.

(a) Describe the motion of the bar as a combination of translation of the centre of mass (CM) and rotation about the CM.

(b) Find the maximum height reached by the CM and the time taken to reach it.

(c) Assuming that the bar passes through its original position after a time t show that $t^2 = 2\pi nl/3g$ where n is an integer.

11. Two objects A and B, each of mass M are fixed to the ends of a rigid rod of negligible mass and length a. The centre of mass, O, of the system is stationary in space, and the system is rotating with angular velocity ω about an axis through O perpendicular to AB. One of the rotating masses (B) strikes a third stationary object C, also of mass M, and the two stick together. The dimensions of the objects themselves may be neglected.



(a) Find the position and velocity of the CM, O′, of the three-particle system just prior to the collision. [Hint: in an obvious notation, $\mathbf{r}_{\rm CM} = (\mathbf{r}_{\rm A} + \mathbf{r}_{\rm B} + \mathbf{r}_{\rm C})/3$; differentiate this to find the velocity of the CM, using the facts that O and C are at rest before the collision. Note that O′ does *not* move as if it were attached to the rod.]

(b) Find the total angular momentum of the system about O′ just before and just after the collision. Hence show that the angular velocity after the collision is $3\omega/4$.

(c) Show that the initial KE of the system is $Ma^2\omega^2/4$ and the final KE is $3Ma^2\omega^2/16$.

(d) How would your results change if the collision between B and C was totally elastic rather than totally inelastic as above? (Assume that A does not aquire instantaneous momentum from the elastic collision of B & C.)

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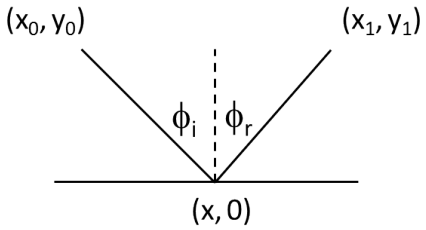
First Year Physics: Prelims CP1

Classical Mechanics: Prof. Armin Reichold

Problem Set VI Lagrangian Dynamics

Questions 1-9 are "standard" examples. Questions 10-16 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in the final 5 lectures of Hilary term.

1. Fermat's principle



A light beam is propagating in the $x-y$ plane in a media whose refraction index n depends only on y.

(a) Use Fermat's principle to show that the trajectory of the beam from (x_0, y_0) to (x_1, y_1) may be obtained by minimizing the functional

$$S(y) = c^{-1} \int_{x_0}^{x_1} n(y)[(1 + y'^2)]^{\frac{1}{2}} dx$$

where $y' = dy/dx$ and c is the speed of light in vacuum.

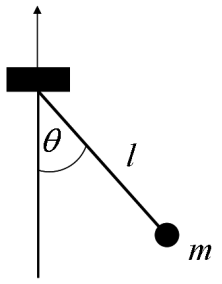
(b) Now let n be independent of x and y. A light ray propagates from (x_0, y_0) to (x_1, y_1) by reflection from the surface of a flat mirror located in the plane $y = 0$ as shown in the figure. Show that the angle of reflection ϕ_r is equal to the angle of incidence ϕ_i.

Euler-Lagrange Equation

2. Motion in two dimensions

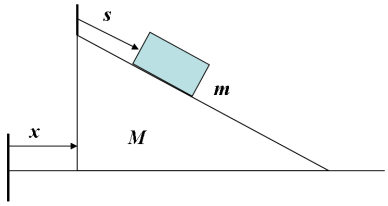
Consider a particle of mass m moving in the (x, y) plane under the influence of the potential $V(r)$ where $\mathbf{r}$ is the postion vector of the particle in an inertial reference frame.
Construct the Lagrangian and the Hamiltonian of the particle in polar coordinates r, θ, hence find which quantities are constants of motion.

3. The simple pendulum



(a) Use the E-L equation to calculate the period of oscillation of a simple pendulum of length l and bob mass m in the small angle approximation.
(b) Now assume that the pendulum support is accelerated in the vertical direction at a rate a, find the period of oscillation. For what value of a the pendulum does not oscillate? Comment on this result.

4. A sliding block

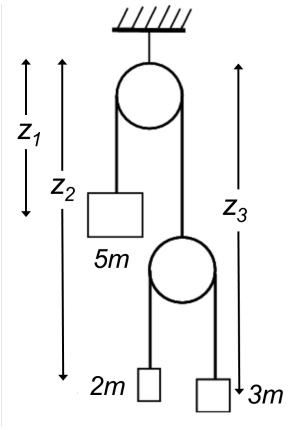


A block of mass m slides on a frictionless inclined plane of mass M, which itself rests on a horizontal frictionless surface.

(a) Choose the displacement of the inclined plane x and the displacement of the block relative to the inclined plane s as generalized coordinates and find the Lagrangian of the system.

(b) Write down the E-L equation for each coordinate and find the acceleration of the inclined plane.

5. Atwood's machine



The three masses shown move in a vertical plane under the influence of constant gravity and the tension in the unextendable strings. Assuming that the pulleys are massless and that all friction forces can be neglected:

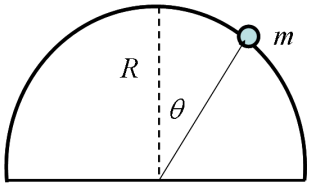
(a) Write down the constraints equation in terms of z_1, z_2 and z_3 that result from the fixed length of the strings, each length L, hence show that the motion of the three masses may be described by two generalized coordinates.

(b) Use the E-L equation to find the acceleration of each mass.

(c) Find the tensions in the two strings; hence show the tension in the upper string is twice that of the lower.

Why does the upper pulley rotate despite the fact that the masses on either side are equal?

6. Particle sliding on a sphere



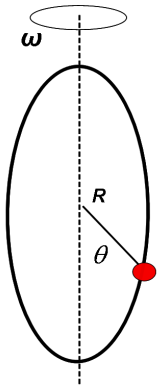
A particle of mass m slides without friction down the surface of a hemisphere of radius R.

(a) Construct the Lagrangian of the problem in terms of the polar coordinates (r, ϑ), in the range when the constraint $r = R$ is valid. Find the equation of motion.

(b) The drawback of imposing the $r = R$ constraint in part (a) is that it does not allow the normal force (i.e. that the hemisphere applies to the particle) to be calculated. To find this force we must introduce the radius of the particle r as an extra parameter (but which only varies by an infinitesimal amount) i.e. rewrite the Lagrangian now with r as a free variable. We must also include in the Lagrangian a potential term $V(r)$ due to the reaction force causing an infinitessimal deformation of the hemisphere. Write down this new Lagrangian and then find the reaction force $F_N = -\frac{\partial V}{\partial r}$ when $r = R$ and $\dot{r} = \ddot{r} = 0$.

(c) Assuming that the particle is released from the top of the sphere from rest, show that the particle leaves the surface at an angle $\cos\vartheta_{max} = 2/3$.

7. A bead on a rotating hoop



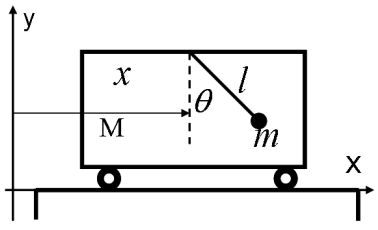
A vertical circular hoop of radius R rotates about a vertical axis at an angular velocity ω. A bead of mass m can slide on the hoop without friction and is constrained to stay on the hoop. By taking the angle ϑ between the radius line and the vertical, as a generalized coordinate:

(a) Find the Lagrangian and the equation of motion in the ϑ coordinate.

(b) Show that there are three equilibrium positions of the bead. Discuss the stabiliy of each equiibrium point and find the frequency of small oscillations about the stable ones.

(c) Find the the total energy $T + V$ and the Hamiltonian of the system. Demonstrate that the Hamiltonian is a constant of the motion but the total energy is not. Why is this?

8. A pendulum with accelerated support



A box of mass M can slide horizontally on a frictionless surface. A simple pendulum of string length l and mass m, is suspended from the ceiling of the box above its centre of mass. Denote the coordinate of the centre of mass of the box by x and the angle that the pendulum makes with the vertical by θ. At $t = 0$ the pendulum displacement is $\theta = \theta_0 \neq 0$

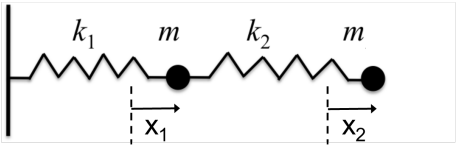
(a) Find the Lagrangian and the equations of motion for the generalized coordinates x and θ.

(b) Find the solutions for x and θ in the small angle approximation, hence show that both the pendulum and the box execute SHO about their centre of mass at an angular frequency

$$\omega = \left[\frac{M+m}{M}\right]^{\frac{1}{2}} \left(\frac{g}{l}\right)^{\frac{1}{2}} .$$

9. Normal modes

Two equal masses m are connected by two massless springs of force constants k_1 and k_2 as shown, and are free to move in the x direction. The system is placed on a horizontal frictionless table and attached to the wall.



(a) Write the Lagrangian of the system using the coordinates x_1 and x_2 that give the displacements of the masses from their equilibrium positions. Use the E-L equations to find the equation of motion of each mass.

(b) Find the solutions in which the two masses execute simple harmonic oscillations at the same frequency (normal modes). What are the common angular frequencies of oscillation for the special cases $k_1 \ll k_2$ and $k_1 \gg k_2$.

Additional questions

10. Revisit Q6 of problem sheet 5.
A compound pendulum consists of a uniform circular disk of radius a and mass m with a series of small holes drilled at regular intervals along a diameter. A horizontal axis about which the disk is free to turn is placed through the hole O', a distance x from the centre of the disk (O).

Find the period T of small-angle oscillations using the Euler-Lagrange equation.

11. A small uniform cylinder of radius a rolls without slipping on the inside of a large, fixed cylinder of radius $b\,(b \geq a)$. Use the E-L equation to show that the period of small oscillations of the rolling cylinder is that of a simple pendulum of length $3(b-a)/2$.

12. Revisit Q3 of problem sheet 5.
A cylindrical reel of thread of radius r and mass m is allowed to unwind under gravity, the upper end of the thread being fixed. (a) Find the initial acceleration of the reel using the E-L equation (b) the initial tension in the thread. [The moment of inertia of a uniform cylinder of radius r and mass m about its axis is $\frac{1}{2}mr^2$.]
[Ans: (a) $2g/3$ m s^{-2}, (b) $mg/3$ N]

13. A ladder of length l and mass m stands on a frictionless floor and leans on a frictionless wall. The ladder is allowed to slide while remaining supported by the floor and wall, and its inclination relative to the vertical wall is given by the angle θ.

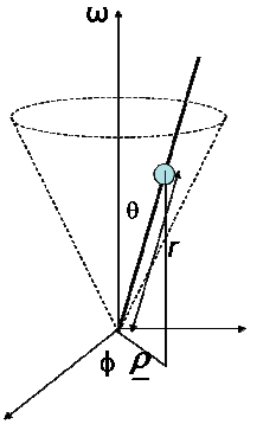
(a) As the ladder slides, show that the centre of mass (CM) of the ladder moves in a circular path, the centre being the point where the floor meets the wall.

(b) Write down the kinetic and potential energy of the ladder. [Hint: you can separate the KE into rotational energy about the CM and transational energy of the CM.] Assume the moment of inertia of the ladder is $ml^2/12$.

(c) Show that the constraints allow you to write the Lagrangian in a single coordinate θ. Determine the equation of motion using the E-L equation.

(d) Use the Lagrangian to discuss the conservation of the total energy E, the Hamiltonian H and the angular momentum J.

14.* Rotating bead



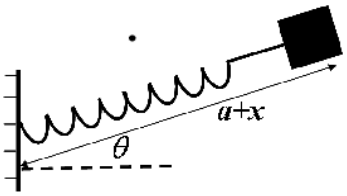
A bead of mass m is constrained to slide on a frictionless wire which is made to rotate about a vertical axis at an angular velocity ω. The wire is tilted away from the vertical by an angle θ and the location of the bead is measured by the coordinate r.

(a) Write down the equation of motion of the bead using the E-L equation. Test the integrity of your equation by taking extreme values of θ.

(b) Find the general solution assuming that at $t = 0, r = r_0, \dot{r} = 0$. Based on this solution, show that for $r_0 = g\cos\theta/\omega^2 \sin^2\theta$, the bead moves in circular motion. Describe the motion for $r < r_0$ and $r > r_0$.

(c) Which of the following quantities is a constant of the bead motion: angular momentum with respect to the origin, the Hamiltonian, total energy?

15. * 2-D spring



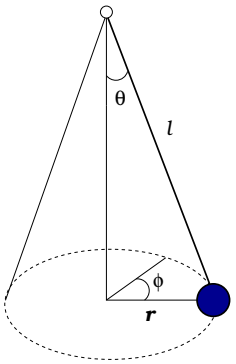
Revisit Q12 of problem sheet 4.
A particle of mass m is attached to the free end of a massless spring of equilibrium length a and spring constant k. The other end of the spring is pivoted to a frictionless horizontal surface and the particle is allowed to move in 2-D in the horizontal plane under the influence of the spring force which is assumed to obey Hook's law.

(a) Write the E-L equations for the polar coordinate (r, θ). Identify the cyclic coordinates and the corresponding conserved quantities. Determine the equation of motion in terms of the variable r.

(b) Write the total energy of the system (for a given angular momentum J) and, using the concept of effective potential, find the radius for circular orbit. Show that it is consistent with the value obtained from Newton's second law.

(c) It was shown in Q12 problem set 4 that if the rest length of the spring is negligible ($a \approx 0$), the path of the particle is elliptical, with r measured from the centre of the ellipse. Use the expression for total energy E to find the major and the minor axes of the ellipse.
[Answer: $a, b = r_0(\sqrt{1 \pm \epsilon})$ where $r_0 = E/k$ and $\epsilon = \sqrt{1 - kJ^2/mE^2)}$]

16. * The Spherical Pendulum



Consider a spherical pendulum which consists of a mass m suspended by a massless string of length l from the ceiling as show below.

(a) Write the Lagrangian of the system in terms of the polar coordinates (θ, ϕ).

(b) Show that ϕ is a cyclic coordinate and find the corresponding conserved quantity. Find the equation of motion in θ and hence show that the pendulum can move steadily around a circle with $\theta = \theta_0$ at an angular frequency Ω given by:

$$\Omega^2 = \frac{g}{l\cos\theta_0}.$$

(c) By relaxing the constraint on l in analogy to Question 6, find the tension in the string. Show that the expression for the tension reverts to that expected from Newton II for circular motion.