1st-year Circuits

Lectures 1 and 2

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Michaelmas 2021

1 Basic definitions

- **Charge:** determines the strength of the electromagnetic force. Measured in Coulombs (C); quantised in units of the electron charge $e = 1.602 \times 10^{-19}$ C.
- **Potential difference:** energy to move unit charge from A to B; $V_{\mathrm{AB}} = V_{\mathrm{B}} - V_{\mathrm{A}} = W/Q = -\int_{\mathrm{A}}^{\mathrm{B}} \boldsymbol{E}.d\boldsymbol{s}$. Measured in volts(V), 1 V = 1 J/C.
- **Current:** rate of flow of charge $I = \frac{dQ}{dt}$. Measured in Amperes (A), 1 A = 1 C/s.
- **Power:** work done per unit time; $P = \frac{VdQ}{dt} = IV$. Measured in Watts, 1 W = 1 J/s.
- **Ohm's Law:** For a component obeying Ohm's law the voltage difference across it is proportional to the current flowing through it; $V = IR$ where R is the resistance measured in Ohms (Ω).
- An **ideal voltage source** maintains a fixed voltage independent of the current it is supplying. The voltage is fixed independent of what is connected to it. A real voltage source can be thought of as an ideal voltage source *in series* with an internal resistance (the smaller the internal resistance the more ideal the voltage source).
- An **ideal current source** supplies a fixed current independent of the voltage required to do so. The current is fixed independent of what is connected to it. A real current source can be thought of as an ideal current source *in parallel* with an internal resistance (the larger the internal resistance the more ideal the current source).

2 Kirchoff's laws

1. **Kirchoff's current law (KCL).** The sum of all currents at a node is zero: $\sum_{\mathrm{node}} I_n = 0$. This is consequence of conservation of charge.
2. **Kirchoff's voltage law (KVL).** Around a closed loop the net change of potential is zero: $\sum_{\mathrm{loop}} V_n = 0$. This results from conservation of energy (in addition to conservation of charge).

3 Passive sign convention

When applying Kirchoff's laws it is easy to get in a muddle with signs, using passive sign convention helps you avoid this. The principle of the sign convention is as follows:

- Voltage and current sources have a '+' sign on the terminal that the current (normally) leaves.
- For all parts of the circuit, choose the direction of current that you are defining as positive.
- For any passive component (e.g. Resistor, Capacitor, Inductor) make a '+' sign on the side of the component that the current is *entering*.
- Now apply Kirchoff's laws. For KVL as you go round the loop a '–' to '+' component has a minus sign and a '+' to '–' component has a plus sign.

4 Methods for solving circuits

In terms of solving simple circuits involving just voltage and current sources and resistors we already have everything we need. However, there are some tricks which can sometimes help you get to the answer more quickly (or with less algebra).

- **Mesh currents:** (i) Label loop currents in all internal loops. (ii) Apply KVL around each loop (taking into account that the current in components that are shared between loops will be the sum or difference of the relevant loop currents). (iii) Solve for the loop currents and use these to obtain the required currents/voltages.
- **Node voltages:** (i) Choose a ground node and label all other nodes. (ii) Apply KCL to each node using the fact that the currents can usually be simply written down as a voltage difference divided by a resistance. (iii) Solve for the node voltages and use these to obtain the required currents/voltages.
- **Superposition.** A linear circuit with more than one voltage/current source can be analysed by considering one source at a time; the total response (i.e. currents or node voltages) of the circuit will be the sum of the individual responses. When analysing just one source, all other voltage sources should be replaced with a wire (short circuit) and all other current sources replaced with a break (open circuit).
- **Thevenin's theorem.** Any linear network of voltage/current sources and resistors can be written as an equivalent circuit of a perfect voltage source (of voltage $V_{\rm eq}$) in series with a resistor ($R_{\rm eq}$). $V_{\rm eq}$ is given by the open-circuit output voltage and $R_{\rm eq} = V_{\rm eq}/I_{\rm eq}$ where $I_{\rm eq}$ is the short-circuit output current.
- **Norton's theorem.** Any linear network of voltage/current sources and resistors can be written as an equivalent circuit of a perfect current source (of current $I_{\rm eq}$) in parallel with a resistor ($R_{\rm eq}$). $I_{\rm eq}$ is given by the short circuit output current and $R_{\rm eq} = V_{\rm eq}/I_{\rm eq}$ where $V_{\rm eq}$ is the open circuit output voltage. Note the simple relationship between the equivalent Thevenin and Norton circuits.

5 Choosing your impedance: measurements and power extraction

- When measuring a voltage the impedance (resistance) of the voltmeter should be $\gg R_{\rm eq}$ where $R_{\rm eq}$ is the equivalent resistance of the circuit (in a Thevenin sense).
- When measuring a current the impedance (resistance) of the ammeter should be $\ll R_{\rm eq}$ where $R_{\rm eq}$ is the equivalent resistance of the circuit (in a Norton sense).
- The maximum power transfer from a power source (with internal resistance $R_{\rm in}$) to a load (with resistance $R_{\rm load}$) will be when $R_{\rm load} = R_{\rm in}$.

1st-year Circuits

Lectures 3 and 4

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Michaelmas 2021

6 Capacitors

- Capacitors store charge. The capacitance C is defined as the charge stored per unit voltage $C = \frac{Q}{V}$; it is measured in Farads (F) or more usually in µF, nF or pF.
- Capacitors in series add as $\frac{1}{C_\mathrm{T}} = \sum_n \frac{1}{C_n}$ and in parallel as $C_\mathrm{T} = \sum_n C_n$.
- The energy stored on a capacitor is given by $W = \frac{1}{2}CV^2$. The energy is stored in the electric field.

7 Inductors

- Inductors oppose a change in current. They are usually coils of wire; a current passing through them results in a linked magnetic field flux Φ; changing the current through the circuit means changing the magnetic flux which (via Faraday's law) results in an induced voltage: $V = \frac{d\Phi}{dt}$ which opposes the change in current. The inductance is defined via $V = L\frac{dI}{dt}$, it tells you the voltage that will be induced if the current is changed at 1 Ampere per second. Inductance is measured in Henrys (H).
- Inductors in series add as $L_\mathrm{T} = \sum_n L_n$ and in parallel as $\frac{1}{L_\mathrm{T}} = \sum_n \frac{1}{L_n}$.
- The energy stored on a inductor is given by $W = \frac{1}{2}LI^2$. The energy is stored in the magnetic field.

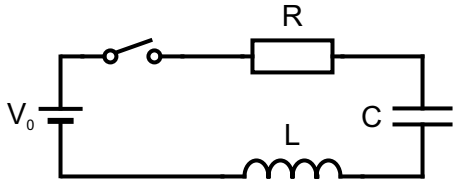
8 Transients in RC and RL circuits

A sudden change in a RC (resisto/capacitor) or RL (resistor/inductor) circuit, such as closing a switch, results in a time-varying transient response. Such problems can be tackled as follows:

- Define current directions and apply the passive sign convention as normal.
- Apply Kirchoff's laws, remembering that for a capacitor $V = \frac{Q}{C}$ and for an inductor $V = L\frac{dI}{dt}$.
- This will lead to a (or a set of) differential equation(s). Solve these equations subject to the appropriate boundary conditions (ask what is the situation at $t = 0$ and at $t = \infty$) taking into account that:
  - it is impossible to change the voltage on a capacitor instantly (because charge has to build up and so that would require infinite current),
  - it is impossible to change the current on a inductor instantly (because that would result in infinite voltage).
- For a series RC and RL circuit the transient response is characterised by a decay/growth of current/voltage with a time constants of $\frac{1}{RC}$ and $\frac{L}{R}$ respectively.

9 LCR series circuit

Consider the circuit below. The capacitor is initially uncharged and at time $t = 0$ the switch is closed.



Applying KVL we find:

$$V_0 = IR + \frac{Q}{C} + L\frac{dI}{dt} \tag{1}$$

where Q is the charge on the capacitor and $I = \frac{dQ}{dt}$ the current in the circuit. This can be re-written as

$$V_0 = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q, \tag{2}$$

or

$$0 = L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I. \tag{3}$$

These are equations for damped simple harmonic motion; note there is a direct equivalence with a mechanical oscillator, with the displacement $x \mapsto Q$, the velocity $v \mapsto I$, the mass $m \mapsto L$ and the spring constant $k \mapsto 1/C$, R defines the damping. The general solution of Eq. (3) is $I = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ where $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ with $\gamma = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$. The nature of the solution depends on γ/ω_0 leading to three different types of solution. In this instance, using the $t = 0$ boundary conditions: $I = 0$ (because of the inductor) and $L\frac{dI}{dt} = V_0$ (because initially $Q = 0$), we have the following solutions:

- Over-damped ($\gamma > \omega_0$)

$$I = \frac{V_0}{2L\sqrt{\gamma^2 - \omega_0^2}}\left(e^{-(\gamma - \sqrt{\gamma^2-\omega_0^2})t} - e^{-(\gamma + \sqrt{\gamma^2-\omega_0^2})t}\right). \tag{4}$$

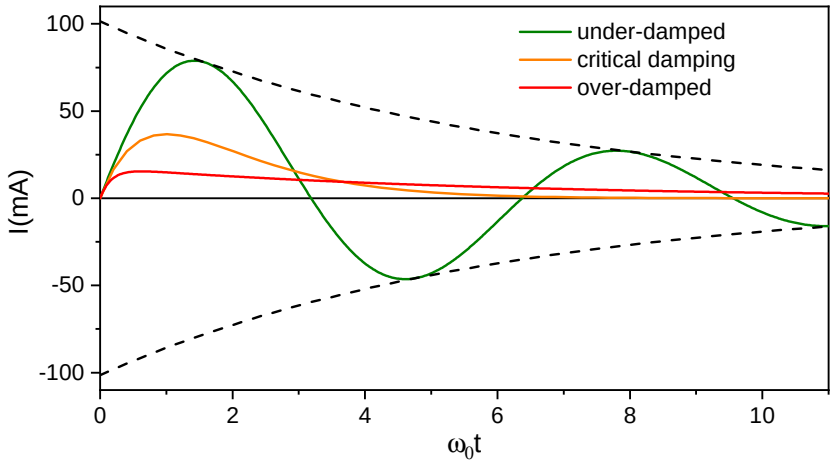
- Critically damped ($\gamma = \omega_0$)

$$I = \frac{V_0}{L}te^{-\gamma t}. \tag{5}$$

- Under-damped ($\gamma < \omega_0$)

$$I = \frac{V_0}{L\sqrt{\omega_0^2 - \gamma^2}}\, e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\; t\right). \tag{6}$$

Example solutions are shown below, where we have chosen $C = 1\,\mu\text{F}$, $L = 10\,\text{mH}$ and $R = 33$, 200 and $600\,\Omega$ leading to a fixed $\omega_0 = 10^4$ rad s^{-1} and $\gamma/\omega_0 = 1/6, 1$ and 3 for the under-, critical- and over-damped cases respectively.



1st-year Circuits

Lectures 5 to 7

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Michaelmas 2021

10 Response of resistors, capacitors and inductors to AC sources

What happens if we apply an time varying current of the form $I = I_0 \cos(\omega t)$ to a resistor, capacitor or inductor?

- The voltage across a resistor will be given by $V_R = IR = I_0 R \cos(\omega t)$.
- The voltage across a capacitor will be $V_C = \frac{Q}{C} = \frac{1}{C}\int I dt = \frac{1}{\omega C} I_0 \sin(\omega t) = \frac{1}{\omega C} I_0 \cos(\omega t - \pi/2)$; the voltage *lags* the current by $\pi/2$ – the capacitor takes time to charge.
- The voltage across an inductor will be $V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t) = \omega L I_0 \cos(\omega t + \pi/2)$; the voltage *leads* the current by $\pi/2$ or the current *lags* the voltage by $\pi/2$ – inductors resist a change in current.

We see that for inductors and capacitors, in addition to the components determining the magnitude of the voltage response they also determine the *phase*. We can encode this information in a *complex impedance* usually denoted Z. As Z is a complex number it can tell us both about the amplitude and phase of the voltage response.

11 Complex impedance

An applied current of $I = I_0 \cos(\omega t)$ can be written as $I = \mathrm{Re}(I_0 e^{i\omega t})$ [we can also write $I_0 \sin(\omega t)$ as $\mathrm{Im}(I_0 e^{i\omega t})$], we can then write the generalised version of Ohm's law $V = \mathrm{Re}(Z I_0 e^{i\omega t}) = |Z| I_0 \cos(\omega t + \phi)$ where Z is the complex impedance which determine both the magnitude and phase of the voltage response.

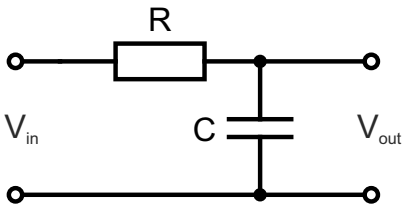
- For a resistor $Z_R = R$.
- For a capacitor $Z_C = \frac{1}{i\omega C}$. Note – it acts like a open circuit as $\omega \to 0$ and a short circuit as $\omega \to \infty$.
- For an inductor $Z_L = i\omega L$. Note – it acts like a short circuit as $\omega \to 0$ and a open circuit as $\omega \to \infty$.
- In a network impedances combine in the same way as resistors: in series $Z_T = \sum_n Z_n$ and in parallel $\frac{1}{Z_T} = \sum_n \frac{1}{Z_n}$.
- The real part of Z is called the resistance (R) and imaginary part the reactance (denoted X).

Note that the fact that we are concerned with the real (or imaginary) parts is usually taken to be implicit such that we often write, for example, $V = IZ$, but it should be remembered that in reality the voltage and currents always have real values. The complex 'currents' and 'voltages', in this approach (which can helpfully distinguished with tilde, i.e $\tilde{V} = \tilde{I}Z$) are called *phasors* and are effectively rotating vectors in the complex plane - the actual voltages and currents are given by the projection of the phasor onto the real (or imaginary) axis. Remembering we are dealing with phasors will become important when we come to consider power in AC circuits.

12 Some example circuits

12.1 Filters

A network of passive components can be used as a filter in which an output voltage ($V_{\rm out}$) across some part of the circuit is a frequency dependent fraction of some input voltage $V_{\rm in}$ such that we can write $V_{\rm in}/V_{\rm out} = f(\omega)$. A simple example of this is the low-pass RC filter shown on the right, which is just a potential divider

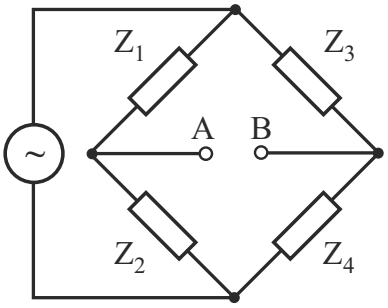


$$\frac{\tilde{V}_{\rm out}}{\tilde{V}_{\rm in}} = \frac{\frac{1}{i\omega C}}{\frac{1}{i\omega C} + R} = \frac{1}{1 + i\omega RC}.$$

The filter is characterized by its amplitude ($|V_{\rm out}/V_{\rm in}|$) and phase ($\mathrm{Arg}(V_{\rm out}/V_{\rm in})$) response which are both functions of ω and usually represented in what are known as Bode plots which show the amplitude and phase response versus ω (often on a log-log plot). Note that we have assumed the input voltage source has no internal impedance and that no current is drawn from the output.

12.2 Bridge circuits

A bridge circuit (as shown to the right) is often used to measure an unknown impedance (e.g. Z_4). Generally, some known impedances (e.g. $Z_{1,2,3}$) are adjusted until the voltage (V_{AB}) across the bridge is nulled ($= 0$). The unknown impedance can be inferred using $V_A = Z_1/(Z_1 + Z_2)$, $V_B = Z_3/(Z_3 + Z_4)$ such that $V_{AB} = 0$ implies



$$\frac{Z1}{Z_1 + Z_2} = \frac{Z_3}{Z_3 + Z_4} \mapsto Z_1 Z_4 = Z_2 Z_3.$$

This expression can be used to find both the real and imaginary parts of the unknown complex impedance.

13 Power dissipation in AC circuits

The power flowing out of a circuit into an electrical component is given by $P = IV$. However, if we are using the complex representation for V and I then when calculating the power we need to be careful to use $P = \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V})$. For AC circuits it is usual to consider the average power dissipated per cycle,

$$\langle P \rangle = \langle \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V}) \rangle = \frac{1}{T} \int_0^T \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V}) dt,$$

where $T = 2\pi/\omega$ is the cycle time. There are various equivalent ways to calculate and express this

$$\langle P \rangle = \langle I_0 \cos(\omega t) \times I_0 |Z| \cos(\omega t + \phi) \rangle = I_0^2 |Z| \left(\cos(\phi) \langle \cos^2(\omega t) \rangle - \sin(\phi) \langle \cos(\omega t) \sin(\omega t) \rangle \right)$$

where we have employed the compound angle formula. Performing the time averages gives:

$$\langle P \rangle = \frac{1}{2} I_0^2 |Z| \cos(\phi) = \frac{1}{2} V_0 I_0 \cos(\phi) = \frac{1}{2} I_0^2 \,\mathrm{Re}(Z).$$

Note that, using the fact that $V_0 = |Z| I_0$ we can also write

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \cos(\phi) = V_{\rm rms} I_{\rm rms} \cos(\phi)$$

where $V_{\rm rms}$ and $I_{\rm rms}$ are the root mean squared values of the voltage and current and $\cos(\phi)$ is sometimes called the power factor. Note that for a sinusoidal voltage (and equivalently current) $V_{\rm rms} = \frac{V_0}{\sqrt{2}}$ (remember that V_0 is the *amplitude* of the voltage) but in general:

$$V_{\rm rms} = \sqrt{\frac{1}{T} \int_0^T V^2(t) dt}.$$

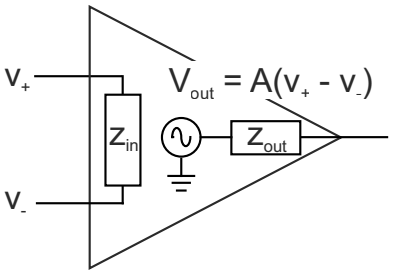
1st-year Circuits

Lectures 8 and 9

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Michaelmas 2021

14 Operational amplifiers



An operational amplifier (or op-amp) is an active (powered) device whose output is given by $V_{\text{out}} = A(V_+ - V_-)$ where A is the open-loop gain and V_+ and V_- are the voltages at he two input terminals. Note, this equation only applies until the outputs *saturate* at the power supply voltages (usually $\pm 15\,\text{V}$). Typically op-amps have a large input impedance Z_{in}, a small output impedance Z_{out} (see diagram) and a large open-loop gain.

14.1 The ideal op-amp and feedback

An *ideal* op-amp is one which has $Z_{\text{in}} = \infty$, $Z_{\text{out}} = 0$ and $A = \infty$. An infinite open-loop gain means that in order for the output not to saturate we require $V_+ = V_-$. To achieve this we apply negative feedback where some of the output is fed back into the (usually $-$) input to ensure $V_+ = V_-$ is a stable point. Whether the feedback is positive or negative can be checked by considering if a small fluctuation of V_{out} changes $V_+ - V_-$ in such a way as to increase or decrease V_{out} – for negative feedback the resulting change will *oppose* the initial fluctuation. For an ideal op-amp operating with negative feedback we can apply the op-amp **GOLDEN RULES**:

1. The inputs draw no current (because $Z_{\text{in}} = \infty$).
2. $V_+ = V_-$ (because $A = \infty$).

14.2 Example circuits

Using the golden rules we can easily solve some simple op-amp circuits. For example, below we have, from left to right:

- A non-inverting amplifier which has a gain of $\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_1+R_2}{R_2}$.
- A inverting amplifier for which $\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{R_2}{R_1}$.
- A integrator for which $V_{\text{out}} = -\frac{1}{RC}\int_0^t V_{\text{in}} dt$.

