CP2 Circuit Theory

Rob Smith robert.smith@physics.ox.ac.uk

https://canvas.ox.ac.uk/courses/117392/pages/cp2-circuit-theory

- Problem set
- Synopsis and reading list
- Lecture summaries
- Slides

Thanks to
Todd Huffman

But, make your own notes because: (i) it is helpful for you to learn, (ii) I will say extra things, (iii) I will write lots of stuff on the blackboard during the lectures

Questions (in both directions)

(Experimentally) using vevox:

Go to vevox.app

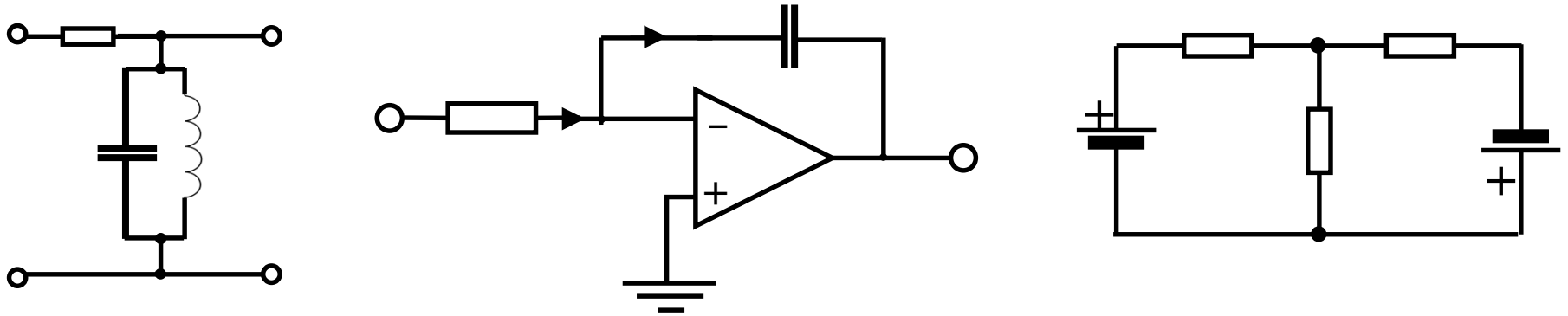
Enter session ID : 123-157-503

Q&A board will be left open

I may sometimes pose questions to you – which you can answer (anonymously).

Aims of this course:

- Understand basic circuit components: resistors, capacitors, inductors, voltage and current sources, op-amps
- Analyse and design simple linear circuits: considering both DC, AC and transient response



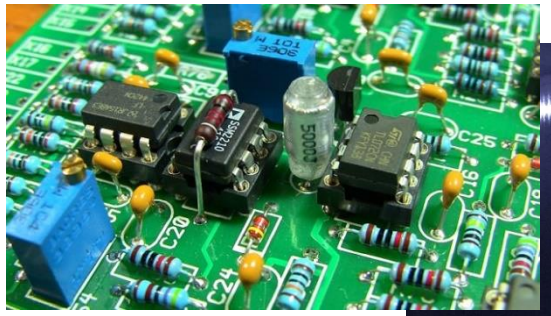
Why study circuit theory?

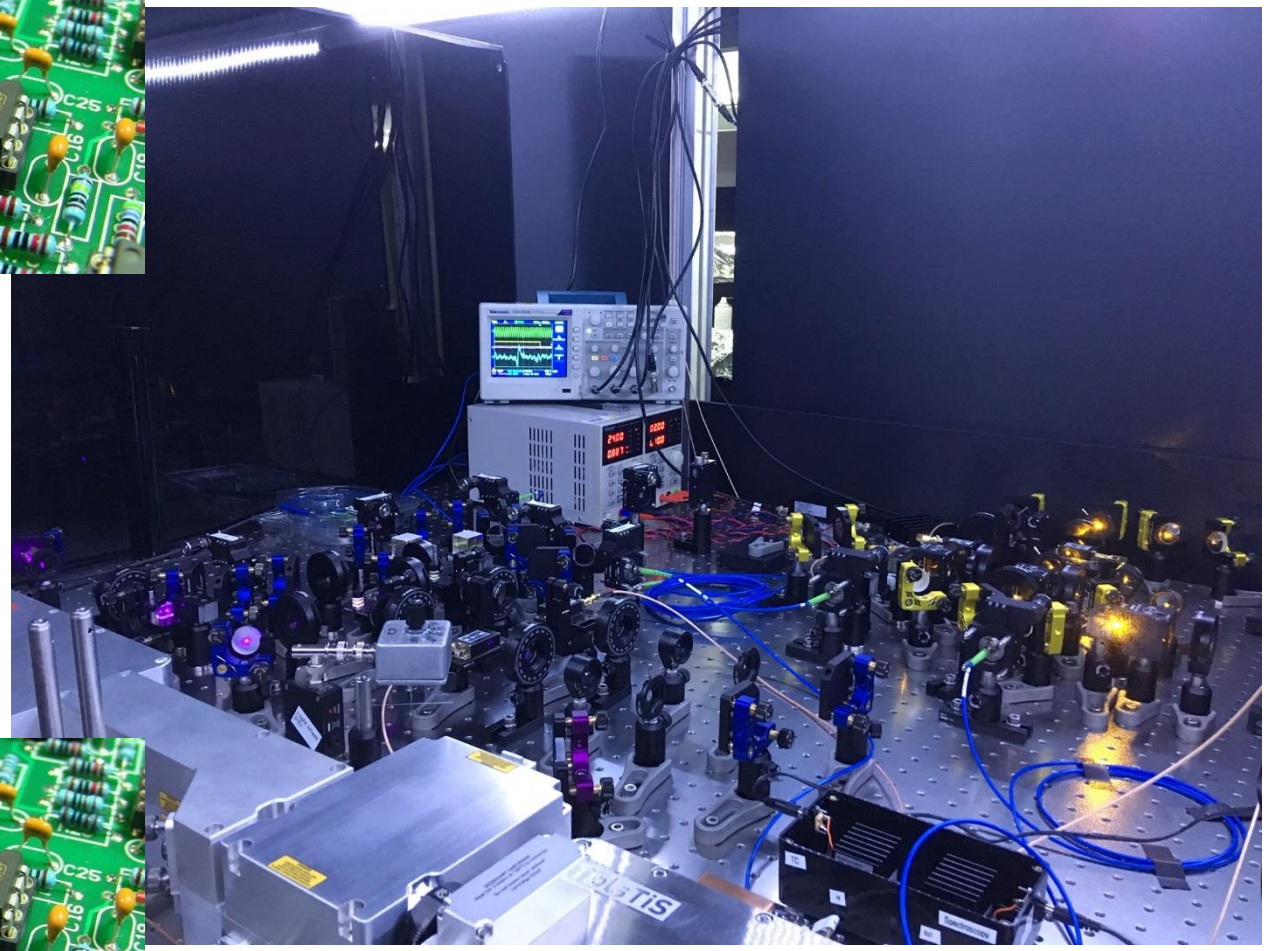
- Foundations of electronics: analogue circuits, digital circuits, computing, communications…
- Scientific instruments: readout, measurement, data acquisition…
- Physics of electrical circuits, electromagnetism, transmission lines, particle accelerators, thunderstorms…
- Not just electrical systems, also thermal, pneumatic, hydraulic circuits, vacuum, control theory

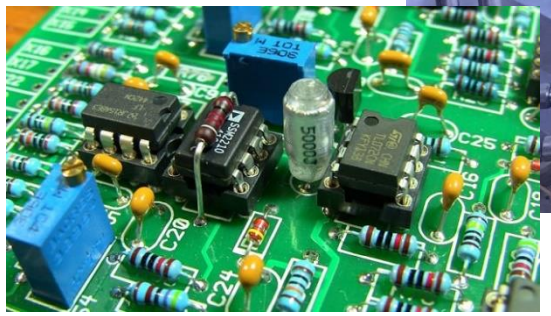
Why study circuit theory?

Control









Readout

Circuit Theory: Synopsis

- Basics: voltage, current, Ohm's law, ideal voltage and current sources…
- Kirchoff's laws and tricks for solving: mesh currents, node voltage, Thevenin and Norton's theorem, superposition…

} Lectures 1&2

- Capacitors and inductors: Stored energy, RC, RL and LCR transient circuits…

} Lectures 3&4

- AC theory: complex notation, phasor diagrams, RC, RL, LCR circuits, resonance, bridges…

} Lectures 5-7

- Op amps: ideal operational amplifier circuits…

} Lectures 8&9

Mathematics required

- Linear differential equations: $\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} = 0$

- Complex numbers: $V(t) = V_0 e^{i\omega t}$, $Z = R + iX$

- Linear equations:

$$I_1 - I_2 - I_3 = 0$$
$$I_1 R_1 + I_2 R_2 = V_1 - V_2$$
$$I_1 R_1 - I_3 R_3 = V_1$$

Covered in CP3/CP4 lectures

Initial definitions:

On blackboard

- Charge: determines strength of electromagnetic force, quantised: $e=1.602\times10^{-19}C$ [C, coulombs]
- Current: rate of flow of charge, $I=\frac{dQ}{dt}(=NAve)$ [A, amps]
- Potential difference: $V=V_A-V_B$, Energy to move <u>unit</u> charge from A to B in electric field [V, volts]
- Power: work done per unit time, $\frac{dW}{dt}=\frac{d(QV)}{dt}=IV$ [W, watts]

Ohm's law (and the resistor)

On blackboard

- Voltage difference $\propto$ current: $V = IR$ $R =$ Resistance [Ω, ohms]
- Symbols





Resistivities	
Silver	1.6×10^{-8} Ωm
Copper	1.7×10^{-8} Ωm
Manganese	144×10^{-8} Ωm
Distilled water	5.0×10^{3} Ωm
PTFE (Teflon)	$\sim10^{19}$ Ωm

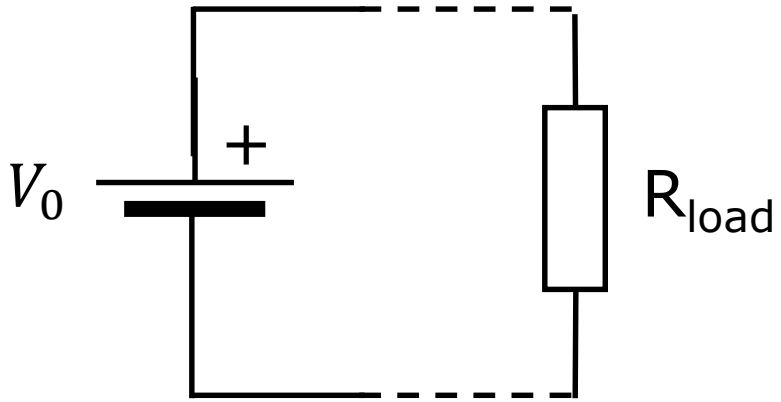
Power dissipated:

$$P = \frac{V^2}{R} = I^2R$$

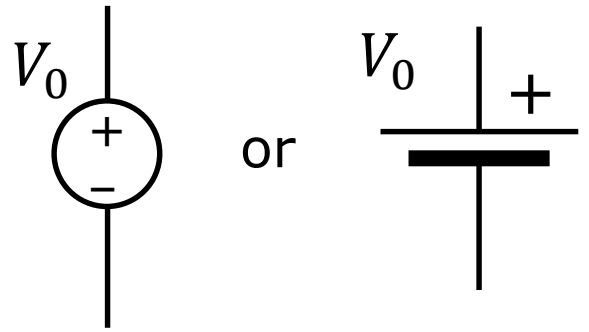
Conductance, $g = \frac{1}{R}$ [seimens] Conductivity, $\sigma = \frac{1}{\rho}$ [seimens/m]

Ideal sources

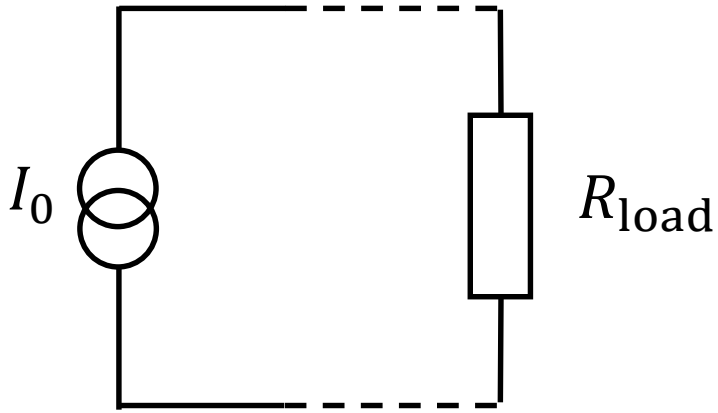
- Ideal voltage source: supplies V_0 independent of current required



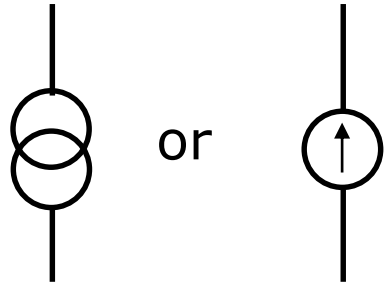
Symbol:



- Ideal current source: supplies I_0 independent of voltage required

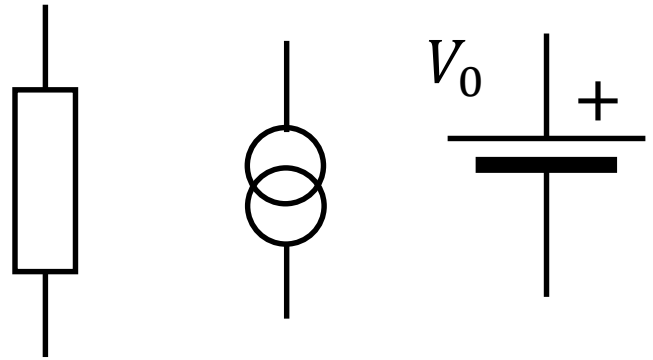


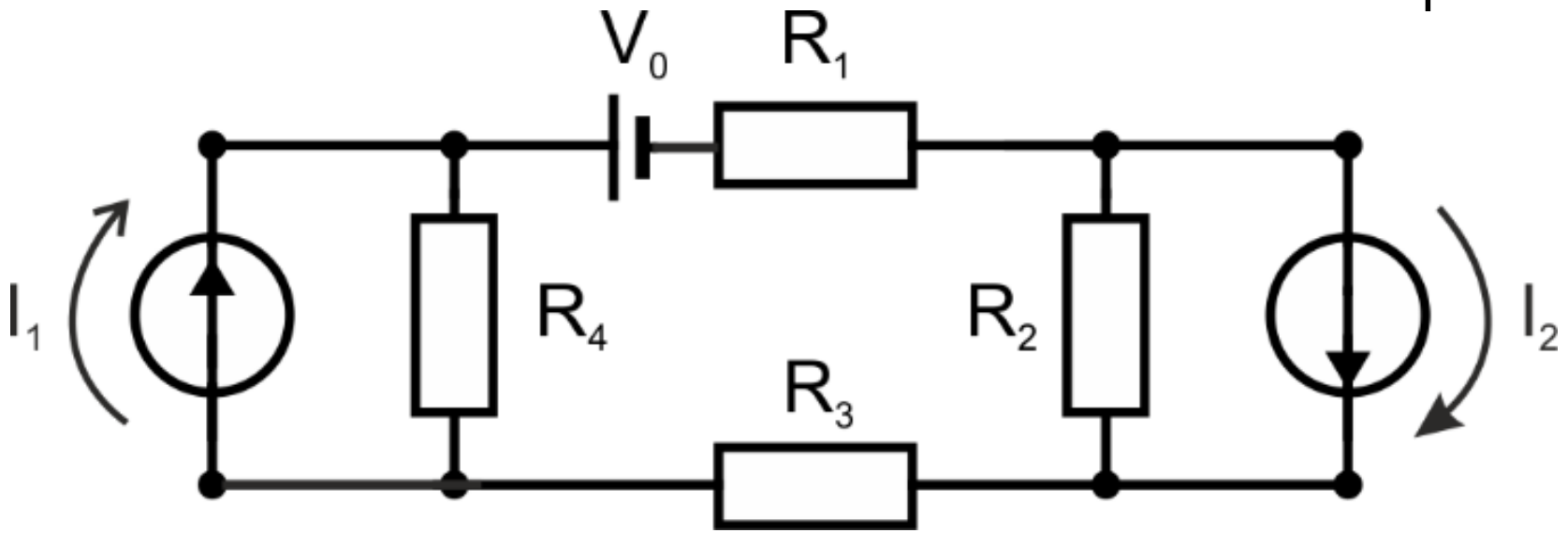
Symbol:



Circuits

- Out of these (3) components we can make (arbitrarily complicated) circuits:





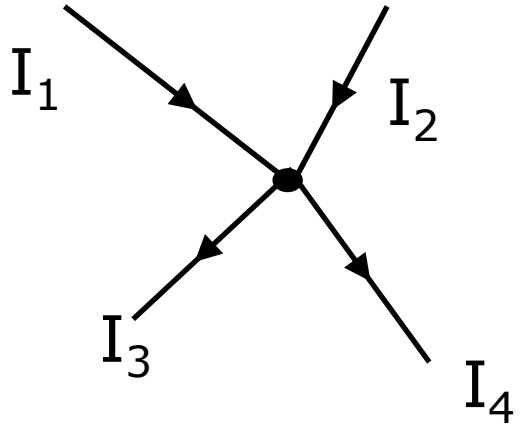
But how do we work out what they do…

Kirchoff's Laws

vevox

On blackboard

- 1, Kirchoff's current law (KCL): Sum of all currents at a node is zero



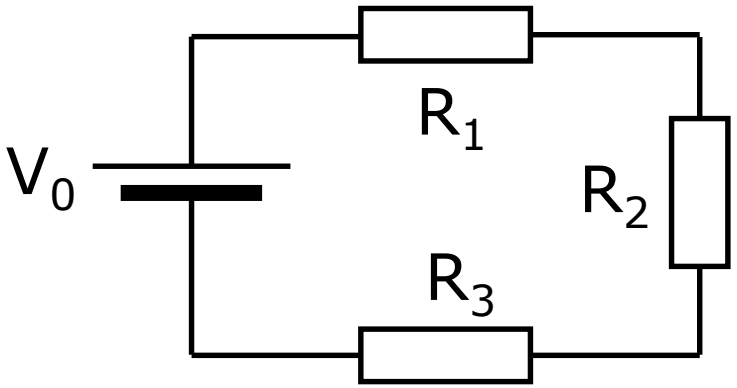
$$I_1 + I_2 - I_3 - I_4 = 0$$

$$\sum_{\text{node}} I_n = 0$$

Doesn't matter whether you pick "entering" or "leaving" currents as positive.
BUT keep the same convention for all currents on one node!

(conservation of charge)

- 2, Kirchoff's voltage law (KVL): Around a closed loop the net change of potential is zero



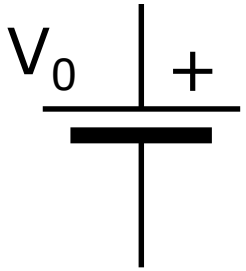
$$\sum_{\text{loop}} V_n = 0$$

But what about the signs of V_n?

(conservation of Energy)

Passive Sign Convention

On blackboard



Sources have a + sign on the terminal the current normally leaves

Where do we put the + sign on a resistor (or other passive component)?



$$V = IR$$

Procedure

- Choose direction of current you are defining as positive.
- For **any passive component** make a + sign on the side of that component that the current is ***entering***.
- When applying KVL, as you go round a loop a – to + component has a negative voltage and a + to – component has a positive voltage.

Learn it; **Live it**; **Love it**!

Passive Sign Convention

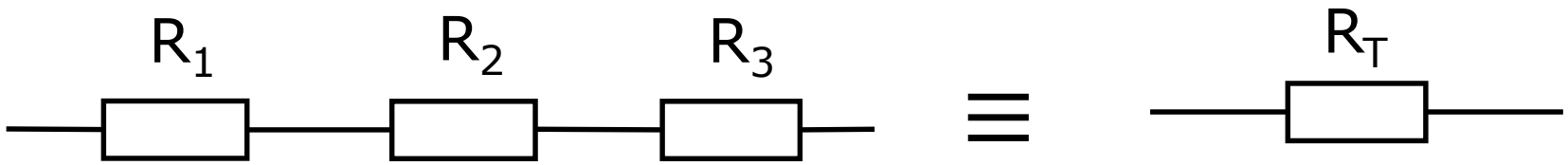
The ‘convention’ is related to how power input/output from a circuit is defined:

- Power flowing out of a circuit into an electrical component is defined as positive.
- Power flowing into a circuit from an electrical component is defined as negative.

Series and parallel circuits

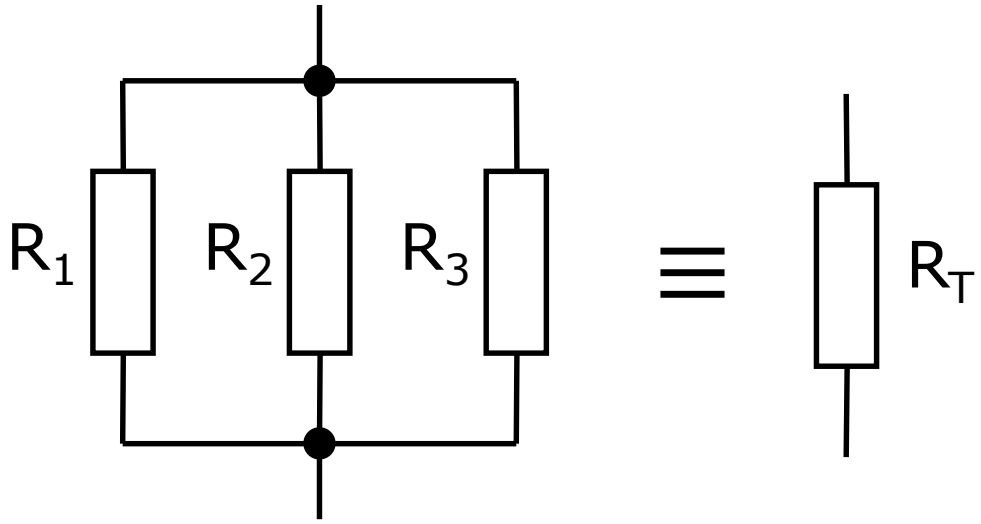
On blackboard

Resistors in series:



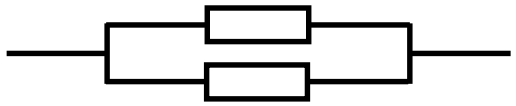
$$R_T = \sum_n R_n = R_1 + R_2 + R_3 + \cdots$$

Resistors in parallel



$$\frac{1}{R_T} = \sum_n \frac{1}{R_n} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

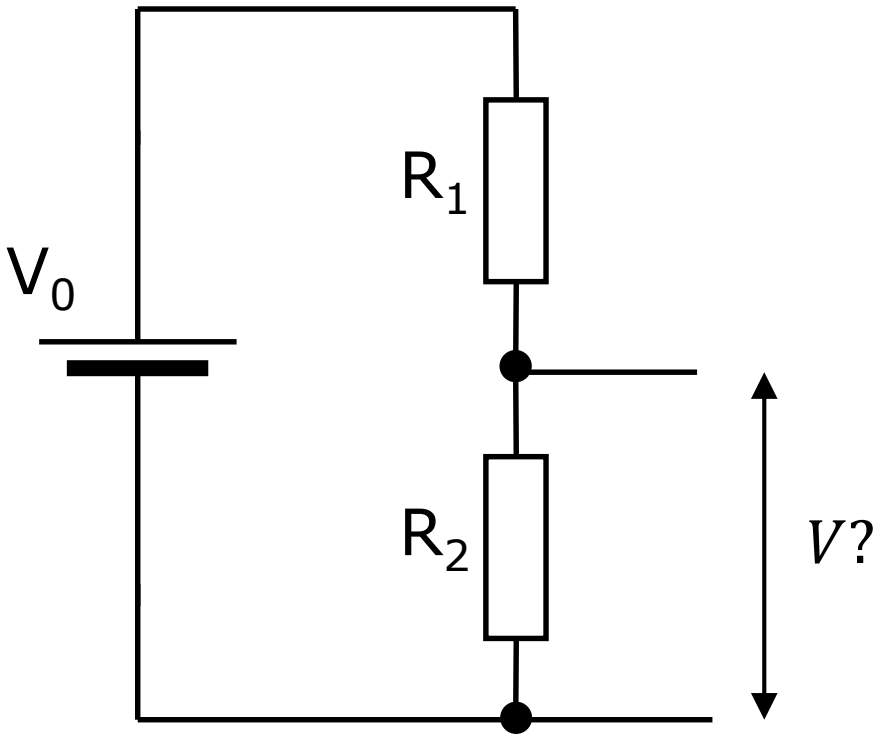
Two parallel resistors:



$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Potential divider

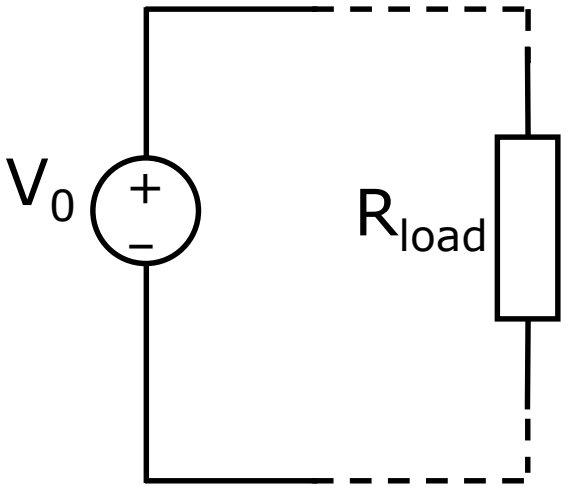
On blackboard



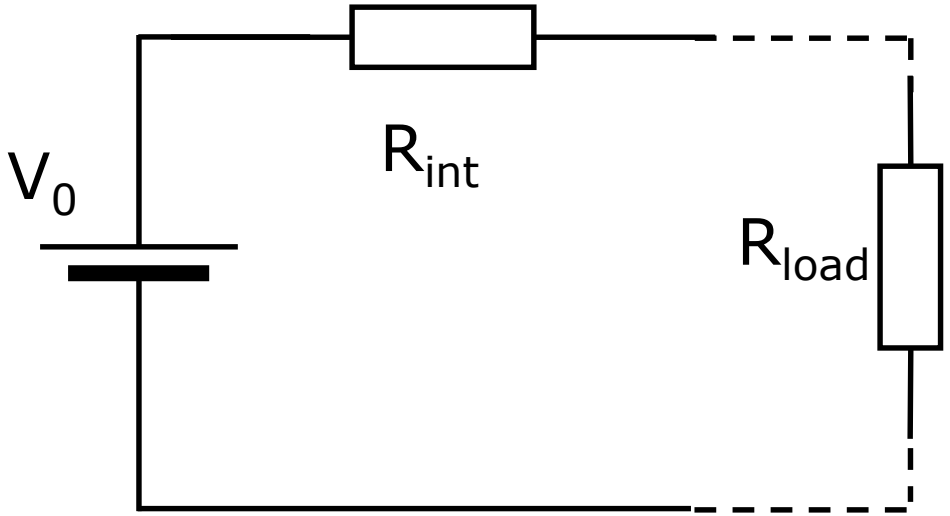
$$V = \frac{R_2}{R_1 + R_2} V_0$$

USE PASSIVE SIGN CONVENTION!!!

Real voltage sources



Ideal voltage source: supplies V_0 independent of current

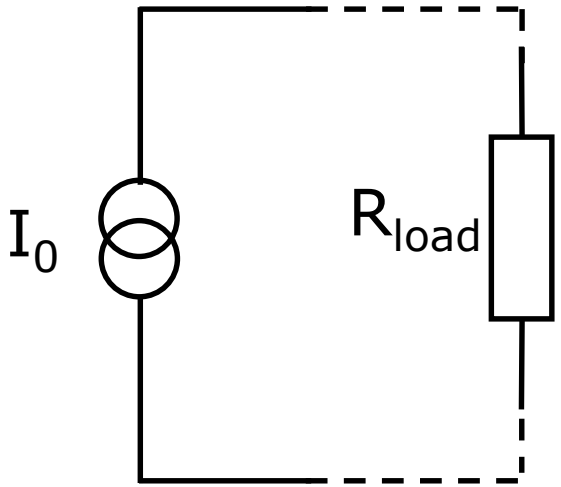


Real voltage source:

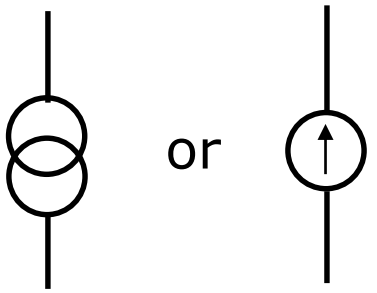
$$V_{\text{load}} = V_0 - IR_{\text{int}}$$

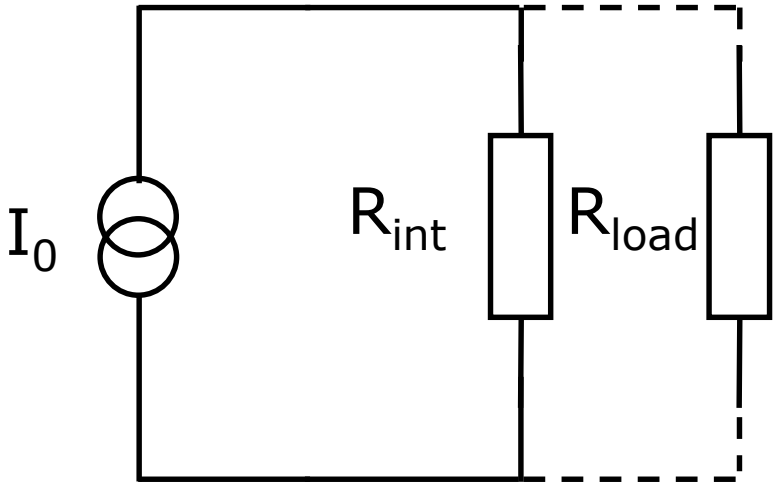
$$V_{\text{load}} = V_0 \times \frac{R_{load}}{R_{\text{load}} + R_{\text{int}}}$$

Real current sources



Ideal current source: supplies I_0 amps independent of voltage

Symbol:  or



Real current source:

$$I_{\text{load}} = I_0 - \frac{V}{R_{\text{int}}}$$

$$I_{\text{load}} = I_0 \times \frac{R_{\text{int}}}{R_{\text{load}} + R_{\text{int}}}$$

Application example

vevox

- You want to measure and record the temperature with a thermistor and a raspberry pi; you also have various different resistors.

- A thermistor is basically a temperature dependent resistor for which the resistance versus temperature is well known. At room temperature it has a resistance of $\mathrm{R_T}{\sim}10\ k\Omega$.

- How do we measure $\mathrm{R_T}$?

Recap

- Ohm’s law and resistance V=IR
- Voltage sources
- Current sources
- Kirchoff’s laws

$$\sum_{\text{node}} I_n = 0 \quad \text{KCL}$$

$$\sum_{\text{loop}} V_n = 0 \quad \text{KVL}$$

- Passive sign convention
- Equivalent resistor networks
- Potential divider

Application example

vevox

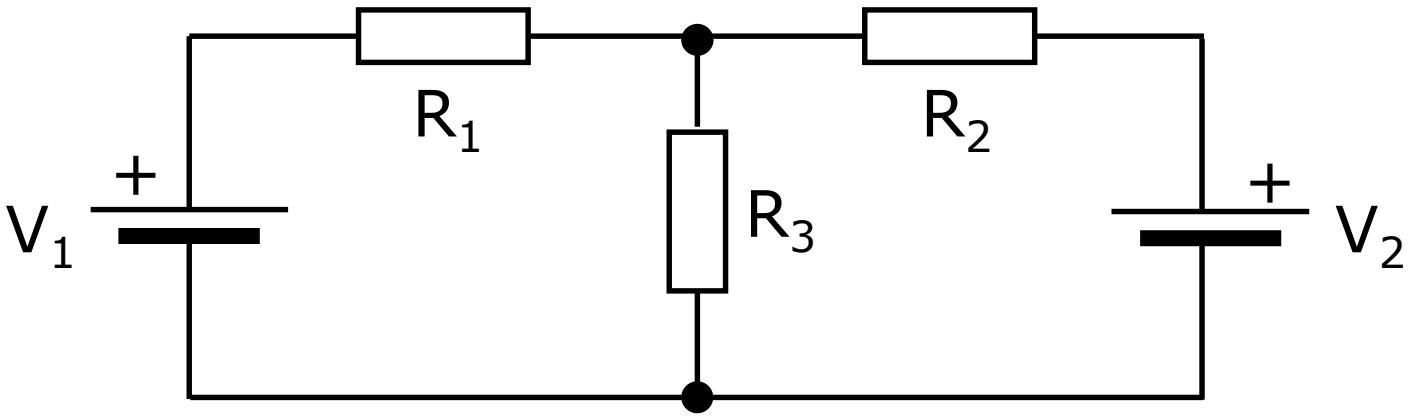
- You want to measure and record the temperature with a thermistor and a raspberry pi; you also have various different resistors.

- A thermistor is basically a temperature dependent resistor for which the resistance versus temperature is well known. At room temperature it has a resistance of $\mathrm{R_T}{\sim}10\ k\Omega$.

- How do we measure $\mathrm{R_T}$?

Example

On blackboard



$V_1 = 9V$
$V_2 = 2V$
$R_1 = 3k\Omega$
$R_2 = 2k\Omega$
$R_3 = 6k\Omega$

This lecture

Methods/tricks for solving circuits

- Mesh currents
- Node voltages
- Superposition
- Thevenin’s theorem
- Norton’s theorem

Measurements and power extraction

- Measuring voltage
- Measuring current
- Power extraction and impedance matching

Mesh currents

On blackboard

1. Label loop currents in all interior loops

2. Apply KVL around each loop

   Using **passive sign convention** and KCL to get currents in shared sections

3. Solve for loop currents and use these to obtain everything else.

Node voltages

On blackboard

1. Choose a ground node and label all other nodes.

2. Apply KCL to each node.
   The currents can usually be directly written down as a $\Delta V/R$.

3. Solve for node voltages and use these to obtain everything else.

Superposition

On blackboard

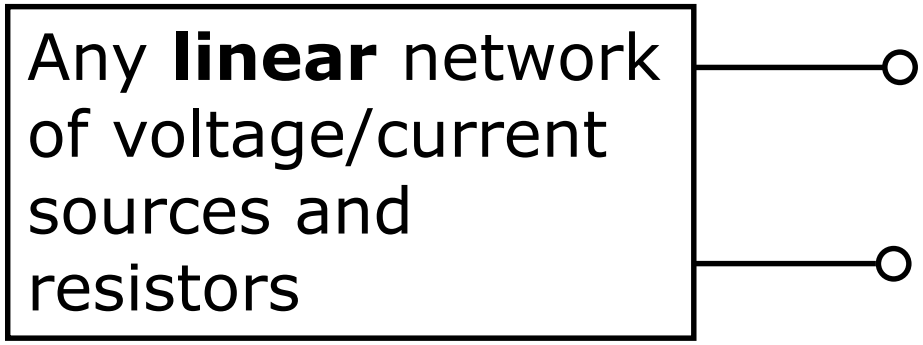
1. A linear circuit with more than one current/voltage source can be analysed by considering one source at a time.

2. When analysing just one source replace other
    - Current sources with a break
    - Voltages sources with a wire

3. The total response of the circuit will be the sum of the individual responses.
    - Make sure you label everything with the same directions

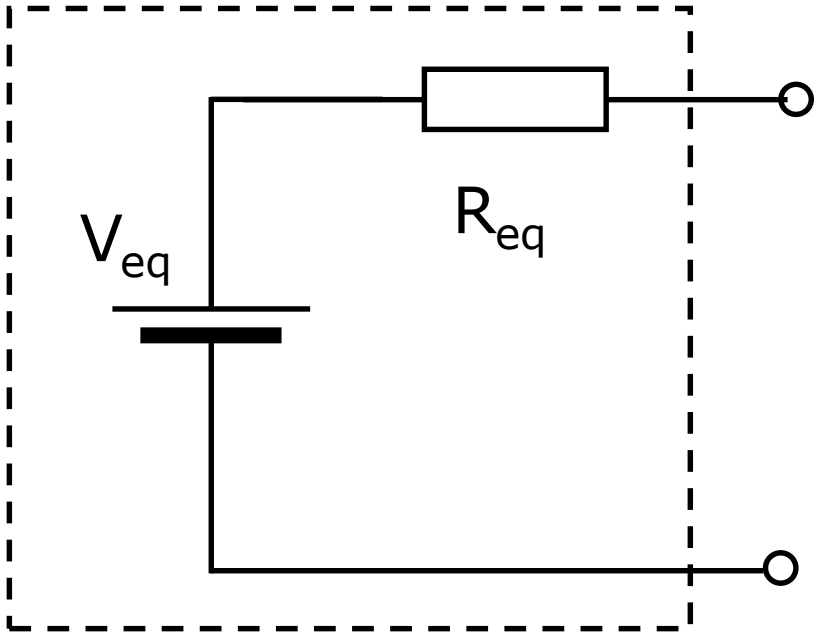
Thevenin's theorem

On blackboard



≡

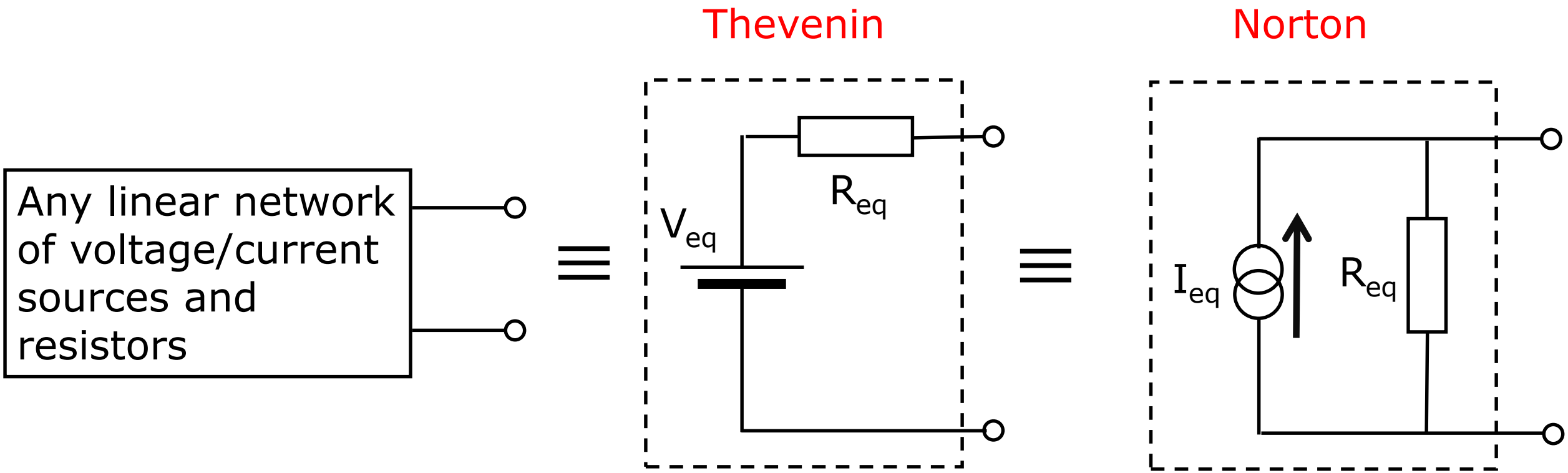
Equivalent circuit



$$V_{\text{eq}} = V_{\text{open-circuit}}$$

$$R_{\text{eq}} = \frac{V_{\text{eq}}}{I_{\text{short-circuit}}}$$

Norton's theorem

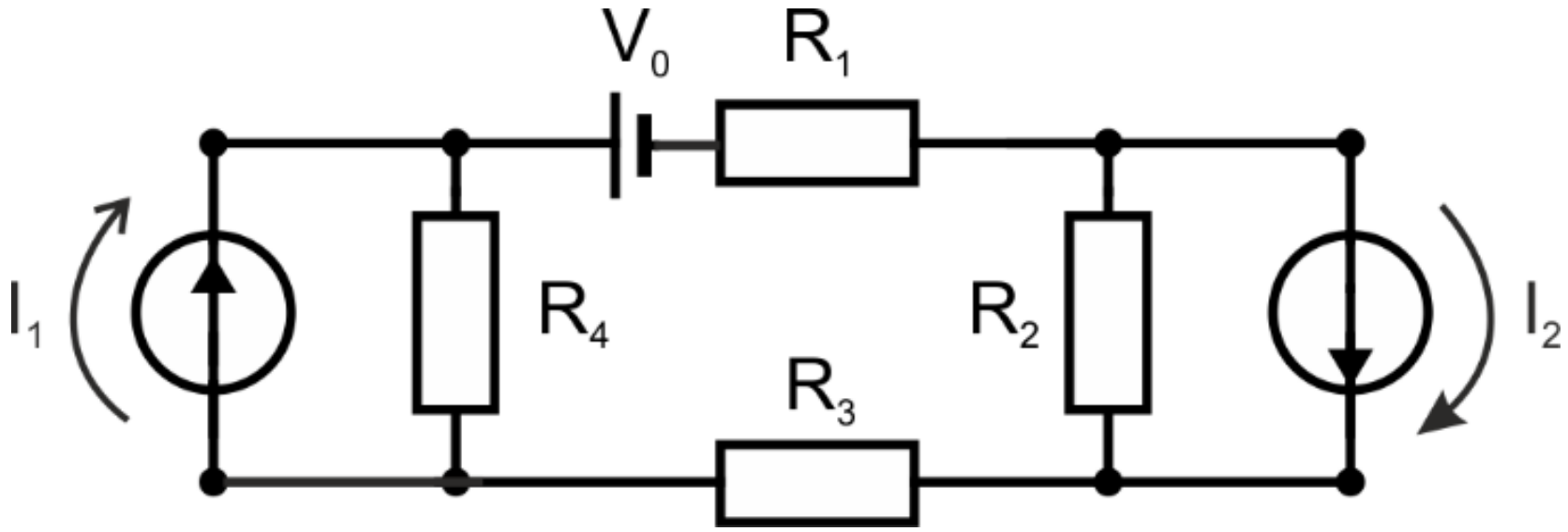


$$I_{\mathrm{eq}} = \frac{V_{\mathrm{eq}}}{R_{\mathrm{eq}}}$$

Methods/tricks for solving circuits

- Mesh currents
- Node voltages
- Superposition
- Thevenin's theorem
- Norton's theorem

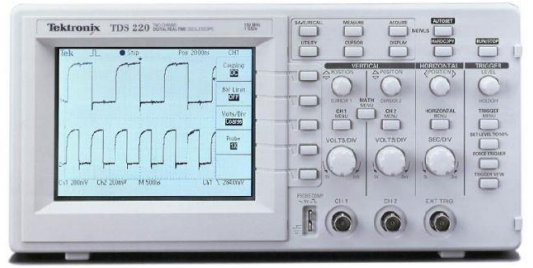
Which trick to use?



Measuring circuit voltages and currents

On blackboard

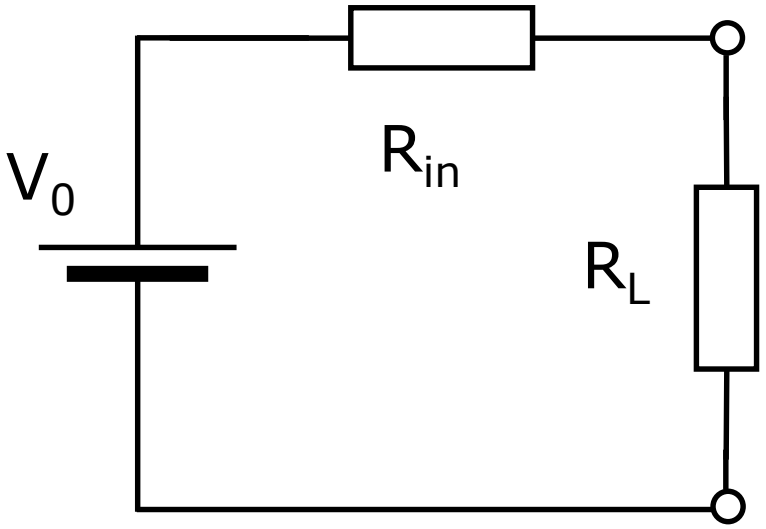
- Voltage measurements: need $R_{in} >> R_{eq}$

10MΩ  1MΩ 

- Current measurement: need $R_{in} << R_{eq}$

Matching: maximum power transfer

On blackboard



Find R_L to give maximum power in load

$$P = \frac{V_L^2}{R_L} = \frac{V_0^2 R_L^2}{R_L(R_{in} + R_L)^2}$$

Max when $\frac{dP}{dR_L} = 0 \quad \Rightarrow R_L = R_{in}$

Maximum power transfer when $R_L = R_{in}$

Note – power dissipated half in R_L and half in R_{in}

Application example

- Stereo system originally has 50W, 8Ω speakers

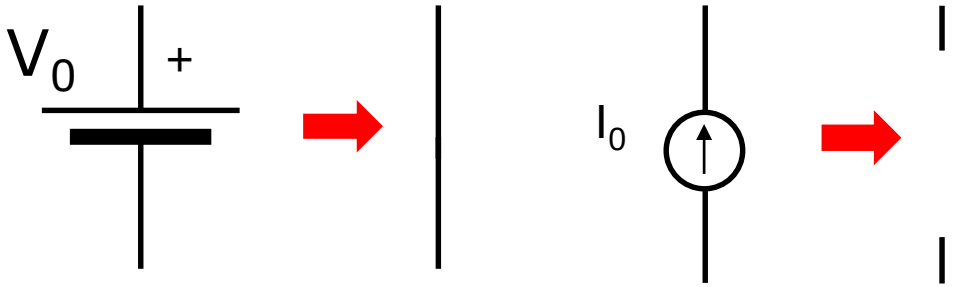
- You are now replacing the speakers from the options:
  - 50W, 4Ω
  - 50W, 16Ω

- Assuming the system was original speakers were well matched which should you choose?



Circuit Theory
Lecture 3

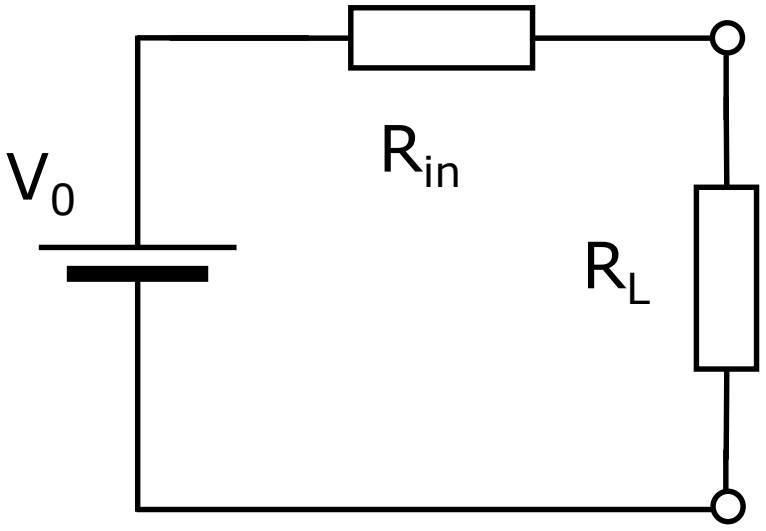
Questions

- Can superposition of voltage sources also be used to find currents separately and add currents together?
- Why 

when doing superposition?
- Could you go through the Norton theorem circuit equivalence?

Matching: maximum power transfer

On blackboard



Find R_L to give maximum power in load

$$P = \frac{V_L^2}{R_L} = \frac{V_0^2 R_L^2}{R_L(R_{in} + R_L)^2}$$

Max when $\frac{dP}{dR_L} = 0 \quad \Rightarrow R_L = R_{in}$

Maximum power transfer when $R_L = R_{in}$

Note – power dissipated half in R_L and half in R_{in}

Recap

- Resistors (V=IR), voltage and current sources
- Kirchoff's laws KCL $\sum_{\text{node}} I_n = 0$ KVL $\sum_{\text{loop}} V_n = 0$ Applying passive sign convention
- Techniques
  - Mesh currents
  - Node voltages
  - Superposition
  - Thevenin theorem
  - Norton theorem

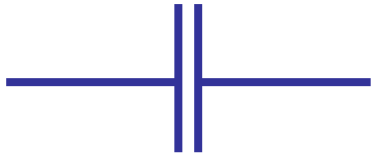
  Choose for situation
- Measuring voltages and currents

  Voltage – high input impedance
  Current – low input impedance
- Impedance matching $R_{\text{in}} = R_{\text{load}}$ maximises power transfer

Today

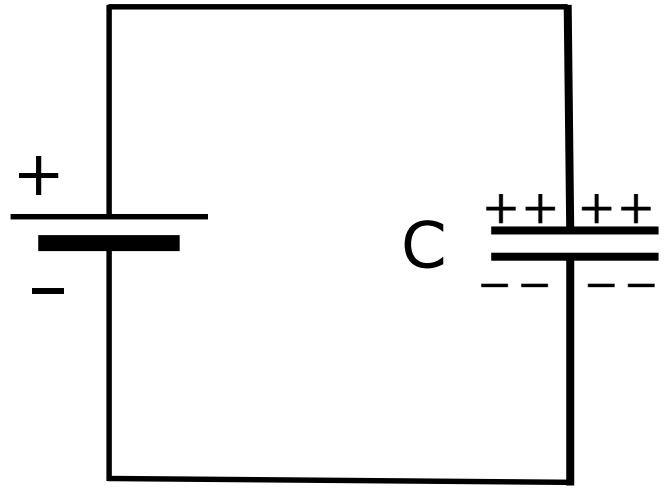
- Capacitors and inductors
- Transient circuits

Capacitors



On blackboard

- Passive components that store charge
- Definition of capacitance: $C = \frac{Q}{V}$
- Units: farad [F]=[CV^{-1}]=[AsV^{-1}]



$$Q = CV \rightarrow \ I = C\frac{dV}{dt}$$

Capacitors oppose a change in voltage

Parallel plate capacitor

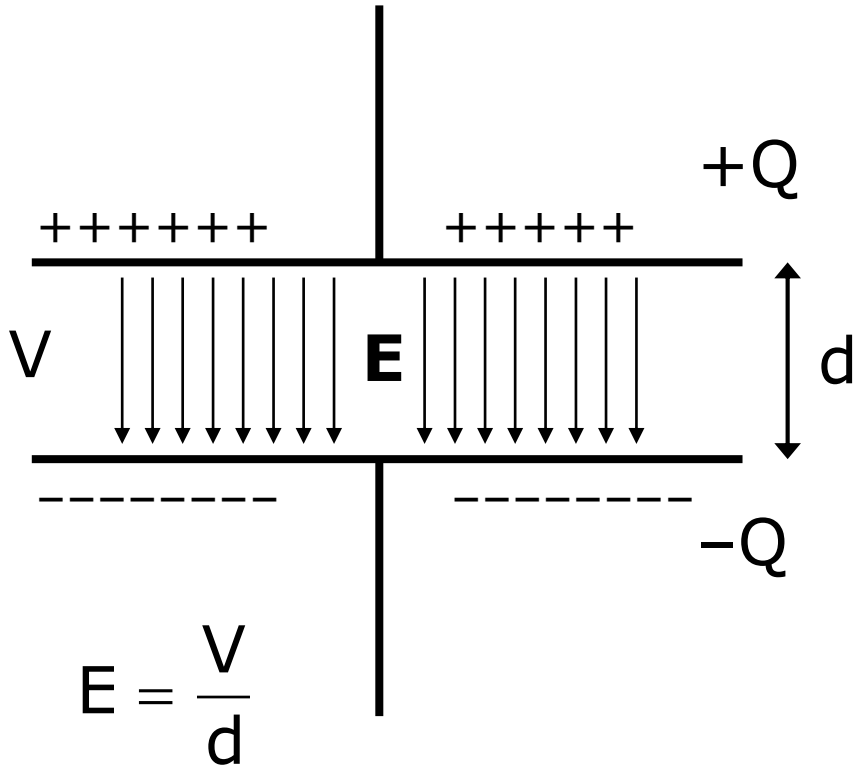


plate area

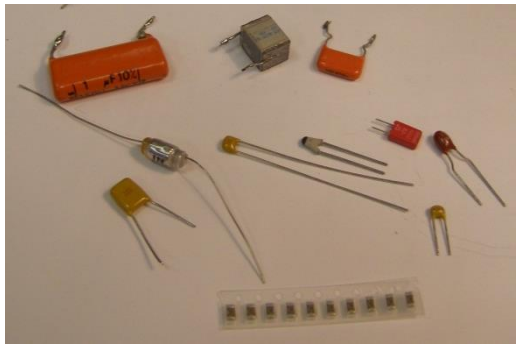
$$C = \frac{\varepsilon A}{d}$$

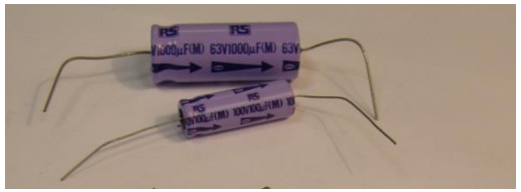
Dielectric permittivity
$\varepsilon=\varepsilon_r\varepsilon_0$

$$\varepsilon_0 = 8.854\times10^{-12} CV^{-1}m^{-1}$$

Capacitor types

		Range	Max voltage
Bidirectional	Ceramic	1pF–1μF	30kV
	Mica	1pF–10nF	500V
	Plastic	100pF–10μF	1kV
Have polarity	Electrolytic	0.1μF–0.1F	500V
	Parasitic	~pF	



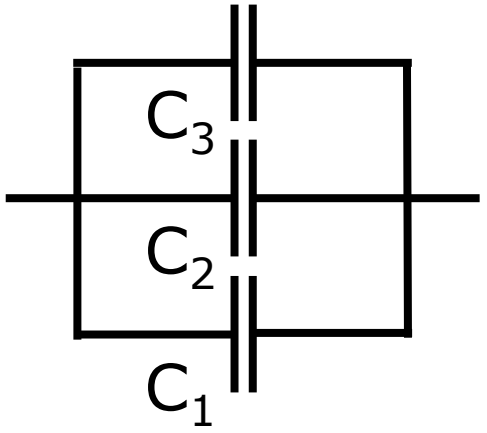


Capacitors in parallel and series

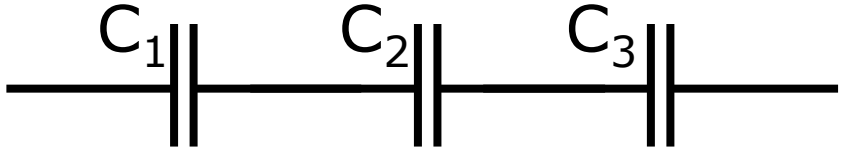
On blackboard

Capacitors in parallel:

$$C_{Total} = \sum_n C_n = C_1 + C_2 + C_3 \ldots$$



Capacitors in series:

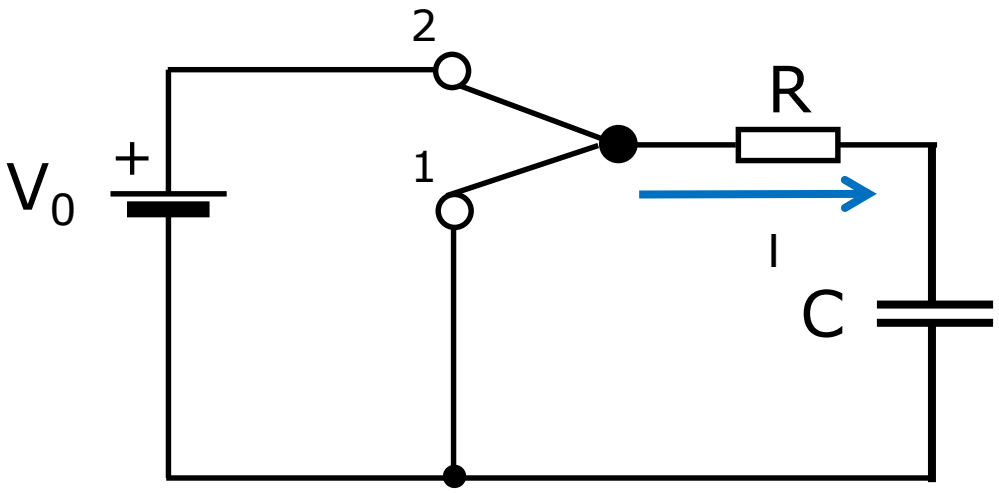


$$\frac{1}{C_T} = \sum_n \frac{1}{C_n} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots$$

Energy stored on a capacitor:

$$W_c = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{Q^2}{2C}$$

RC circuits



- Switch in Position "1" for a long time.
- Then instantly flips at time t = 0.
- How do we analyse this?

and Apply KVL

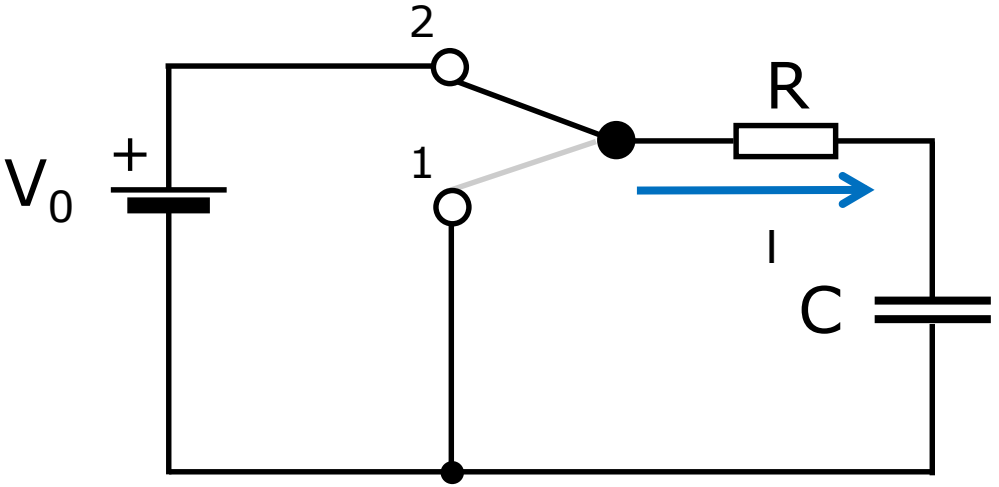
ALWAYS start by asking these three questions:

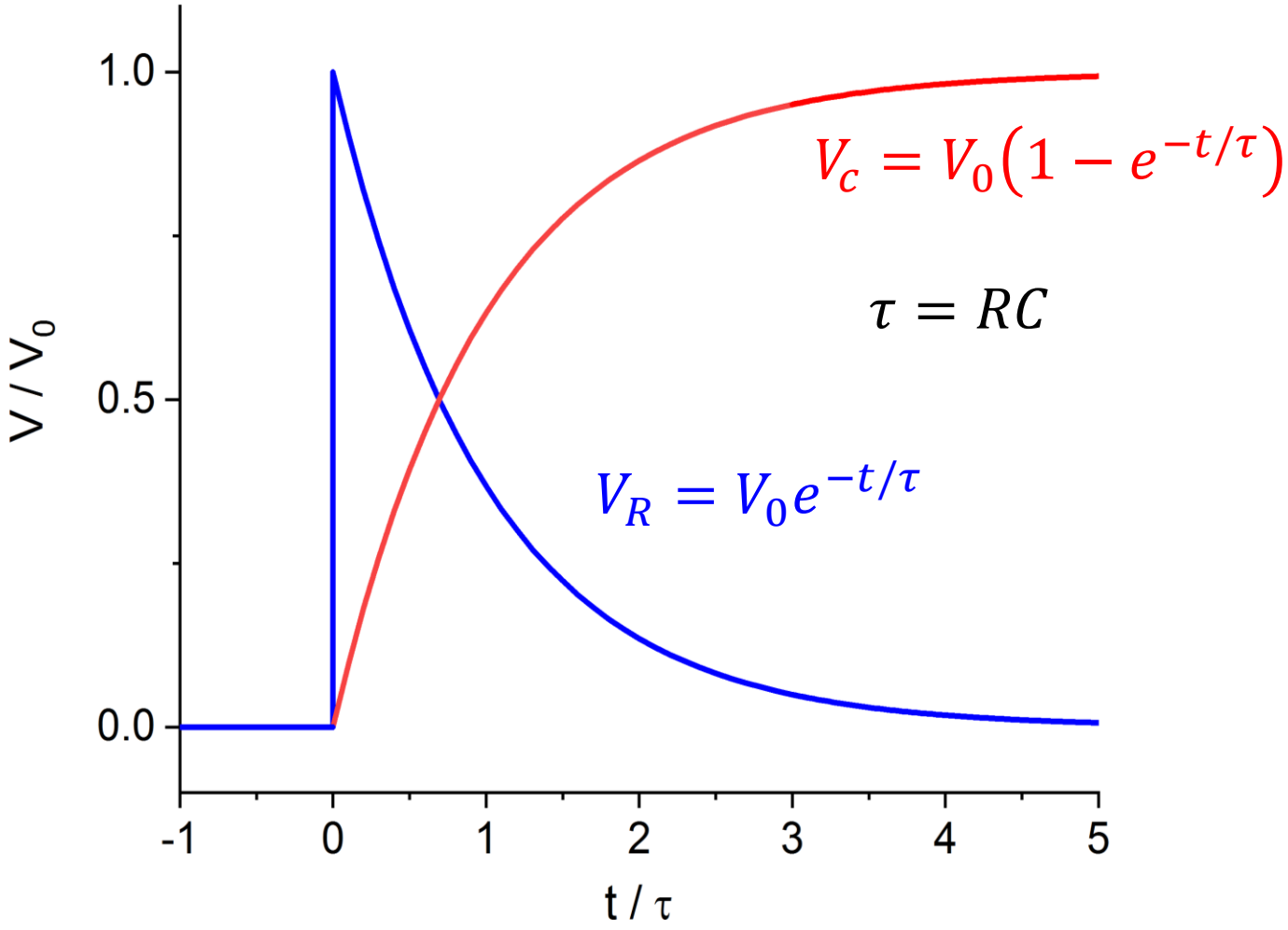
1. What does the Circuit do up until the switch flips? (Switch has been at pos. 1 for a VERY long time. easy)
2. What does the circuit do a VERY long time AFTER the switch flips? (easy)
3. What can we say about the INSTANT after switch flips? (easy if you know the rule)

It is impossible to change the voltage on a capacitor instantly!

RC circuits

On blackboard





$$V_c = V_0\left(1 - e^{-t/\tau}\right)$$

$$\tau = RC$$

$$V_R = V_0 e^{-t/\tau}$$

Inductors

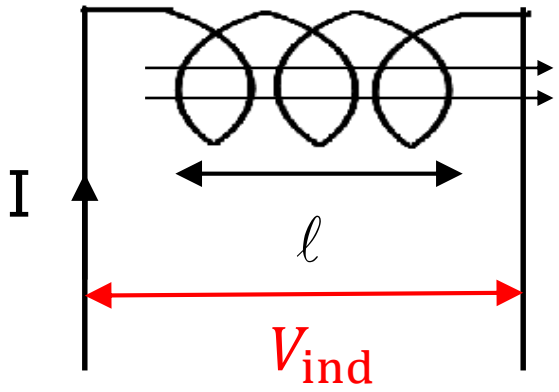




On blackboard

- Oppose a change in current

Solenoid
N turns



$$B = \frac{\mu_0 NI}{l}$$

$$\Phi = \int B.dA = BAN = \frac{\mu_0 N^2 A}{l} I$$

Try to change I:

Electromagnetic induction
- Faraday's law;
- Lenz's law - V_{ind} will oppose change

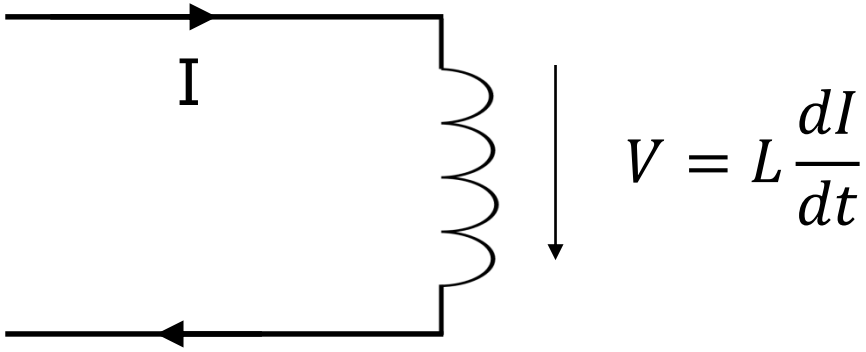
$$V_{\text{ind}} = \frac{d\Phi}{dt} = L\frac{dI}{dt}$$

inductance $$L = \frac{\mu_0 N^2 A}{l}$$

Units: henrys [H] = [VsA^{-1}]

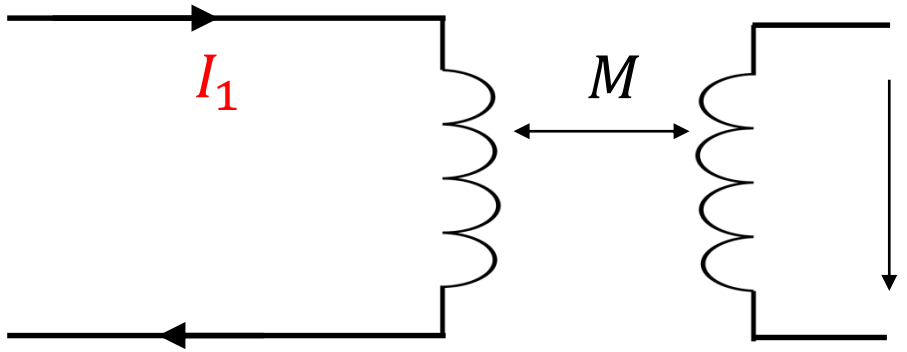
Mutual inductance

- Self Inductance



$$V = L\frac{dI}{dt}$$

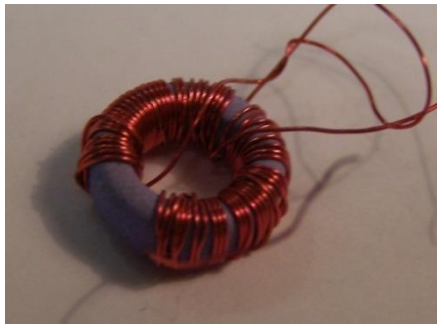
- Mutual Inductance



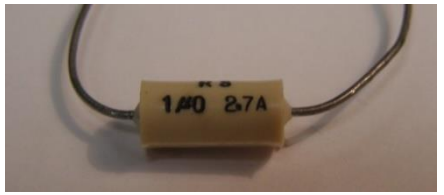
$$V_2 = M\frac{dI_1}{dt}$$

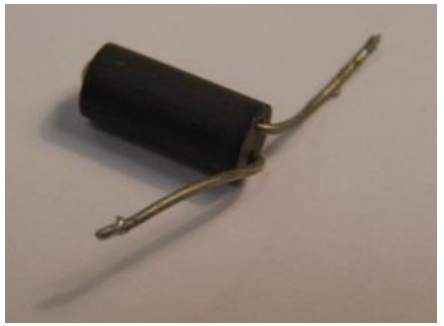
Inductors

- Wire wound coil with air core
- Wire wound coil with ferrite core
- Wire loops
- Straight wire

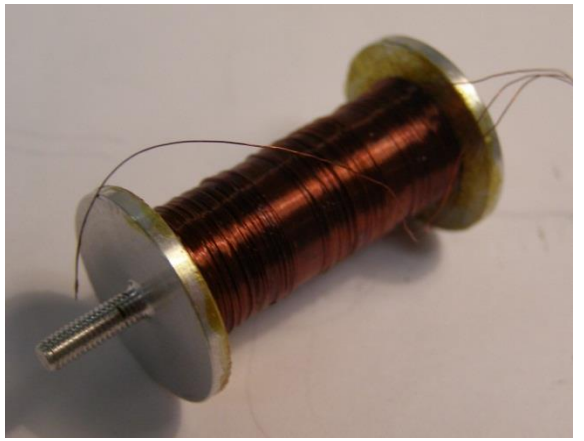


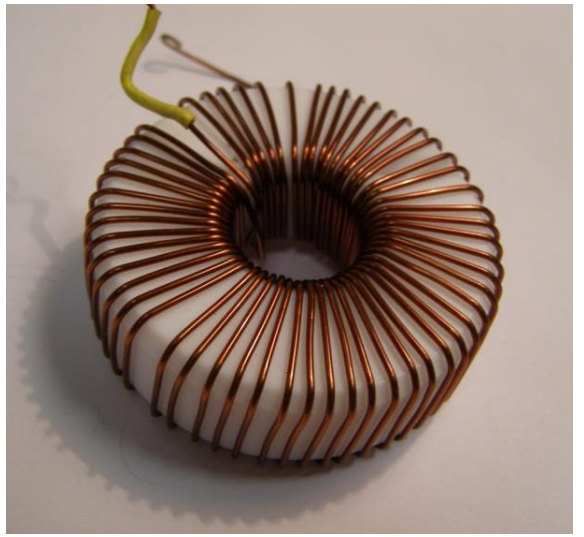










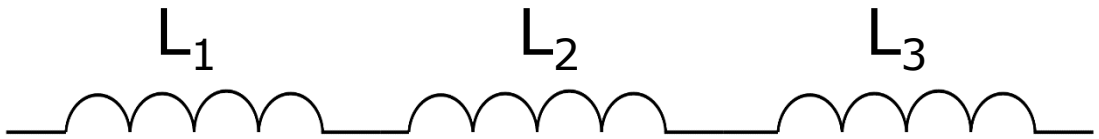




Inductors

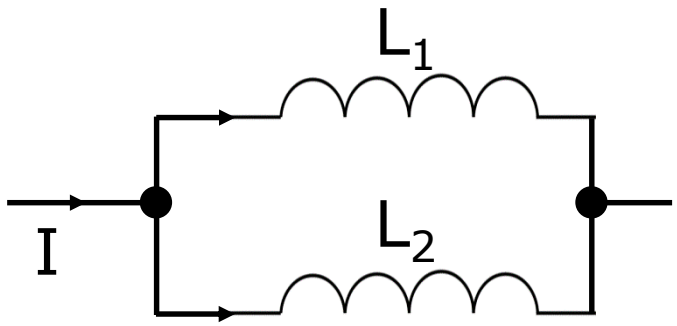
On blackboard

Inductors in series:



$$L_T = \sum_n L_n$$

Inductors in parallel:



$$\frac{1}{L_T} = \sum_n \frac{1}{L_n}$$

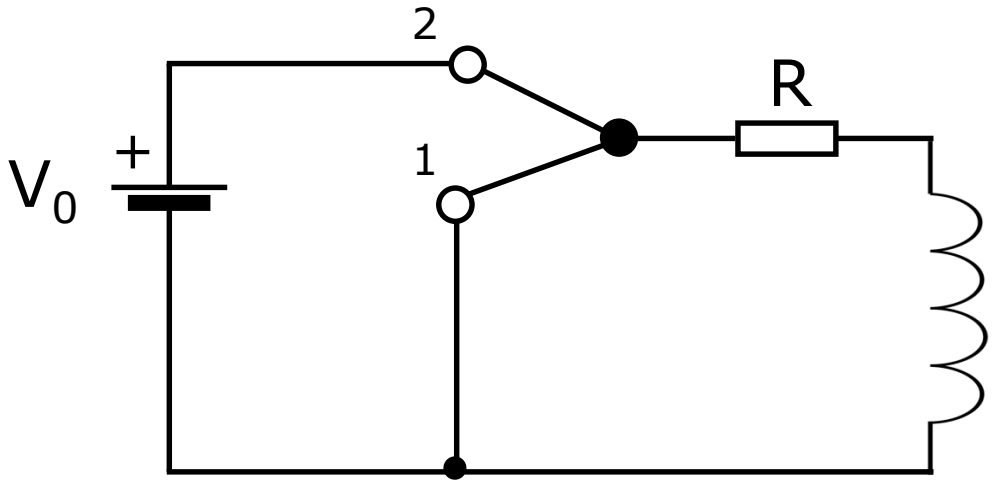
Energy stored on a inductor:

$$W_L = \frac{1}{2} L I^2$$

Energy stored in the magnetic field

Circuit Theory
Lecture 4

RL circuits



- Switch in Position "1" for a long time.
- Then instantly flips at time t = 0.
- How do we analyse this?

and Apply KVL

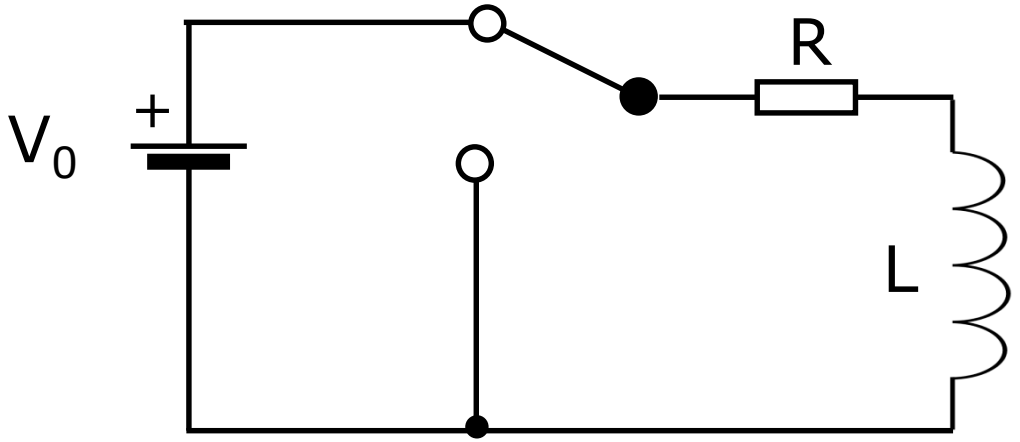
ALWAYS start by asking these three questions:

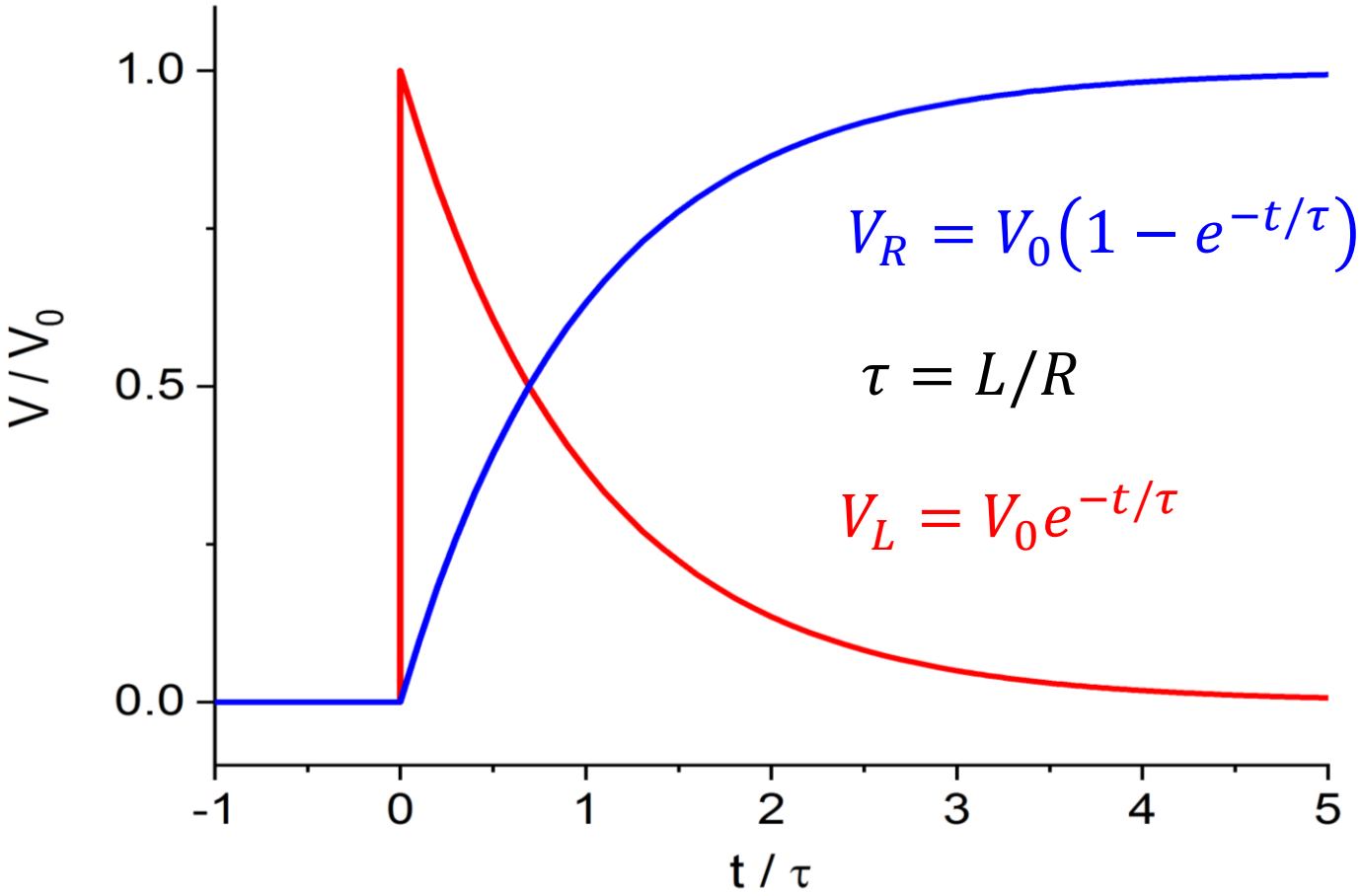
1. What does the Circuit do up until the switch flips?
   (Switch has been at pos. 1 for a VERY long time. easy)
2. What does the circuit do a VERY long time AFTER the switch flips? (easy)
3. What can we say about the INSTANT after switch flips? (easy if you know the rule)

It is impossible to change the current on a inductor instantly!

RL circuits

On blackboard





$$V_R = V_0\left(1 - e^{-t/\tau}\right)$$

$$\tau = L/R$$

$$V_L = V_0 e^{-t/\tau}$$

Recap

- Capacitors store charge; they resist a change of voltage

$$V = \frac{Q}{C} \qquad I = C\frac{dV}{dt} \qquad W = \frac{1}{2}CV^2$$

It is impossible to change the voltage on a capacitor instantly!

- Inductors oppose a change in current

$$V = L\frac{dI}{dt} \qquad W = \frac{1}{2}LI^2$$

It is impossible to change the current on a inductor instantly!

- Transient circuits

$$\tau = RC$$

Capacitor and resistor in series

$$\tau = \frac{L}{R}$$

Inductor and resistor in series

Today

- LC circuit

- LCR circuit

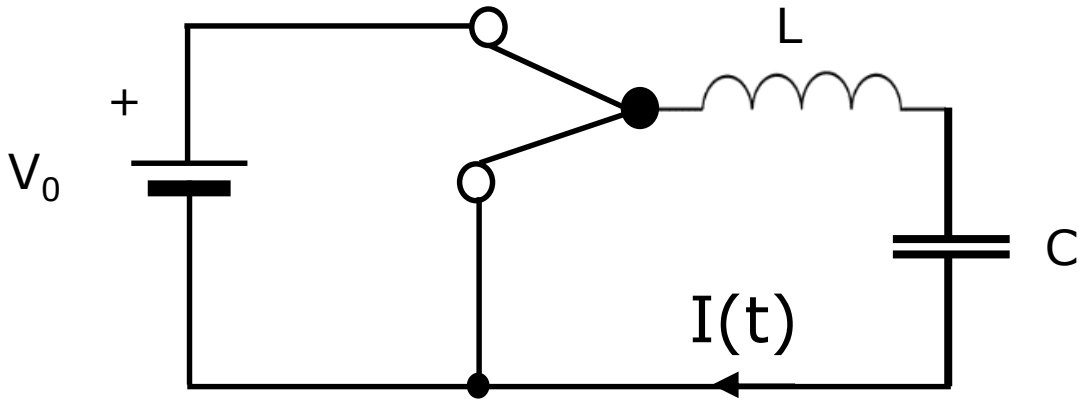
LC circuit

On blackboard

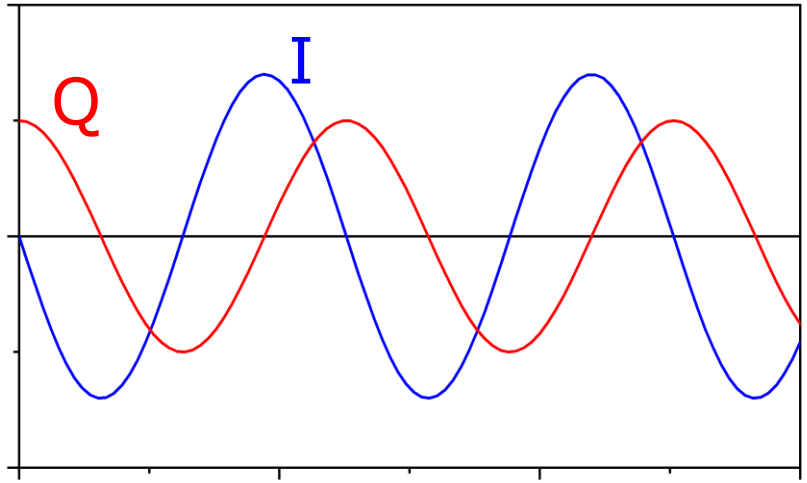
At $t = 0^+$: $Q = Q_0 = CV_0$ & $I = 0$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0 \quad \rightarrow \quad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \quad \rightarrow \frac{d^2Q}{dt^2} + \omega_0^2 Q = 0$$

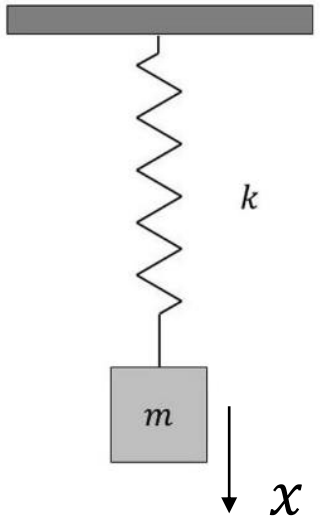
$$\omega_0 = \sqrt{\frac{1}{LC}}$$



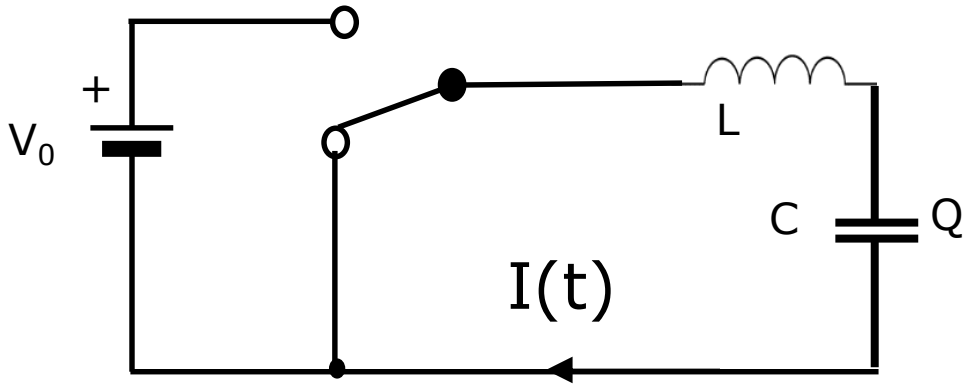
$$\rightarrow \quad Q = Q_0 \cos \omega_0 t \quad \rightarrow \quad I = -Q_0 \omega_0 \sin \omega_0 t$$



Harmonic oscillator



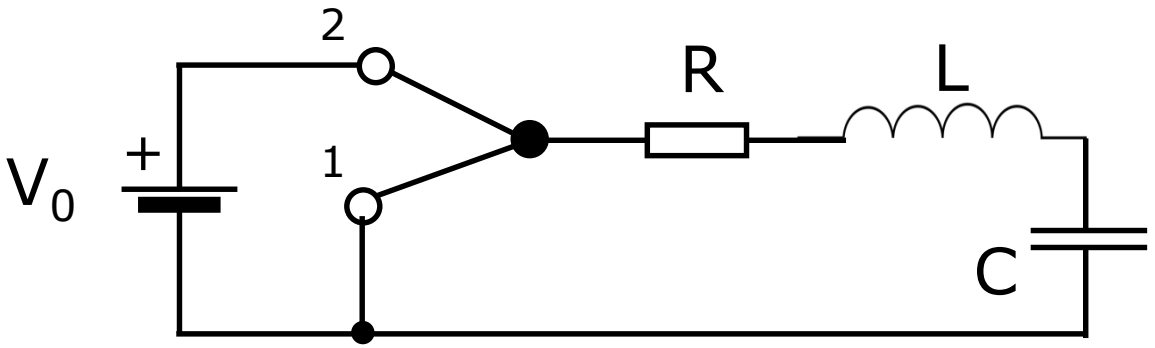
equivalent



$$x \rightarrow Q$$
$$\dot{x} \rightarrow I$$
$$m \rightarrow L$$
$$k \rightarrow \frac{1}{C}$$

LCR circuit

On blackboard



Initial conditions

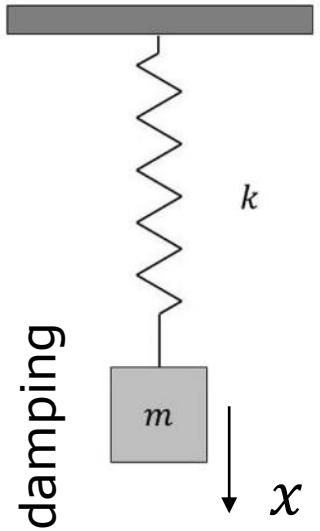
It is impossible to change the current on a inductor instantly!

$$I(t = 0^+) = 0 \quad \rightarrow \quad \mathrm{V_R} = 0$$

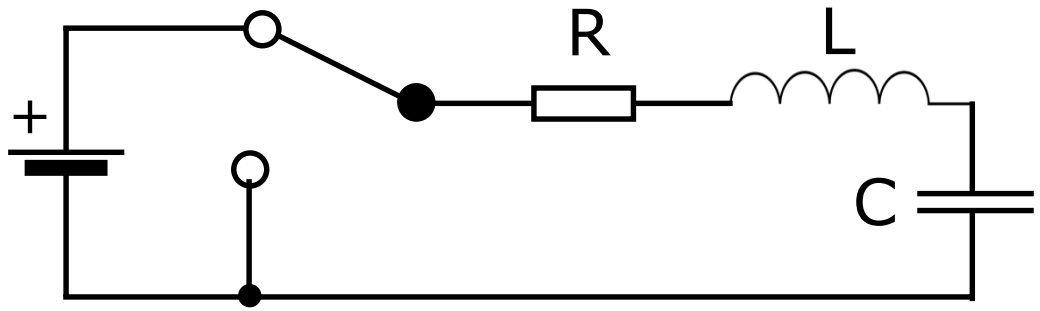
It is impossible to change the voltage on a capacitor instantly!

$$Q = 0 \quad \rightarrow V_c = 0 \rightarrow V_L = \; L\frac{dI}{dt}\Big|_{t=0^+} = V_0$$

Damped harmonic oscillator



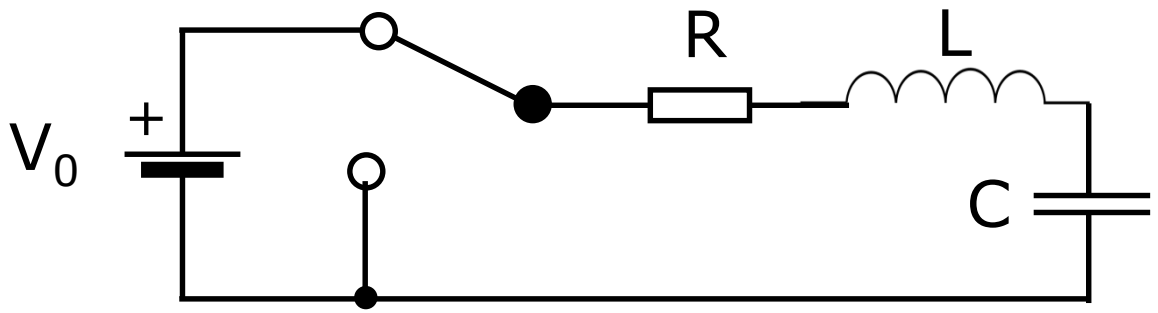
equivalent



$$x \to Q$$
$$\dot{x} \to I$$
$$m \to L$$
$$k \to \frac{1}{C}$$
$$b \to R$$

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

LCR circuit



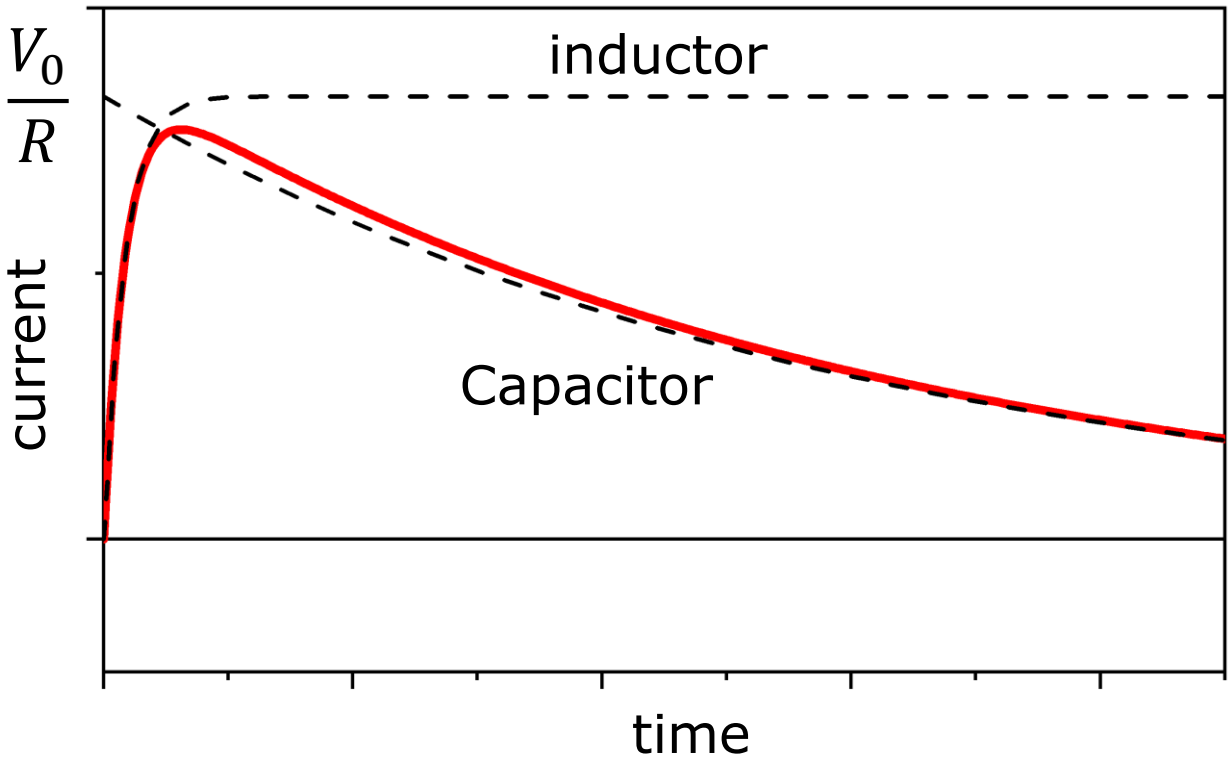
At $t = 0^+$: $Q = 0$ & $I = 0$ & $L\frac{dI}{dt} = V_0$

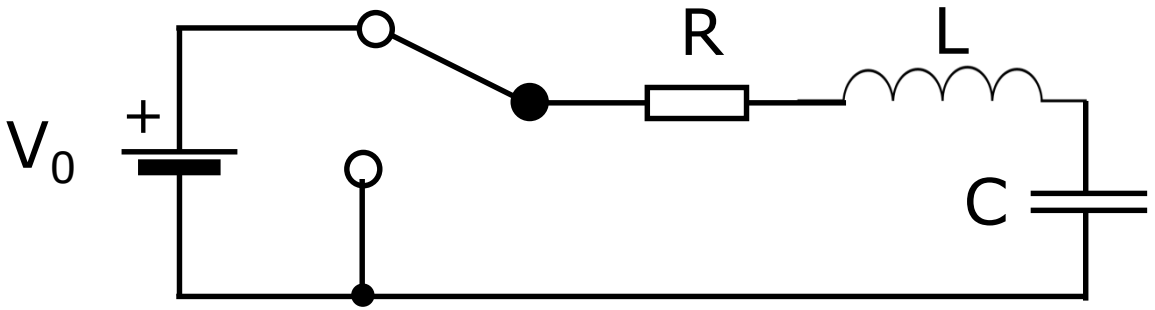
$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

$$\frac{d^2I}{dt^2} + 2\gamma\frac{dI}{dt} + \omega_0^2 I = 0 \qquad \gamma = \frac{R}{2L} \qquad \omega_0^2 = \frac{1}{LC}$$

Over-damped $(\gamma > \omega_0)$

On blackboard





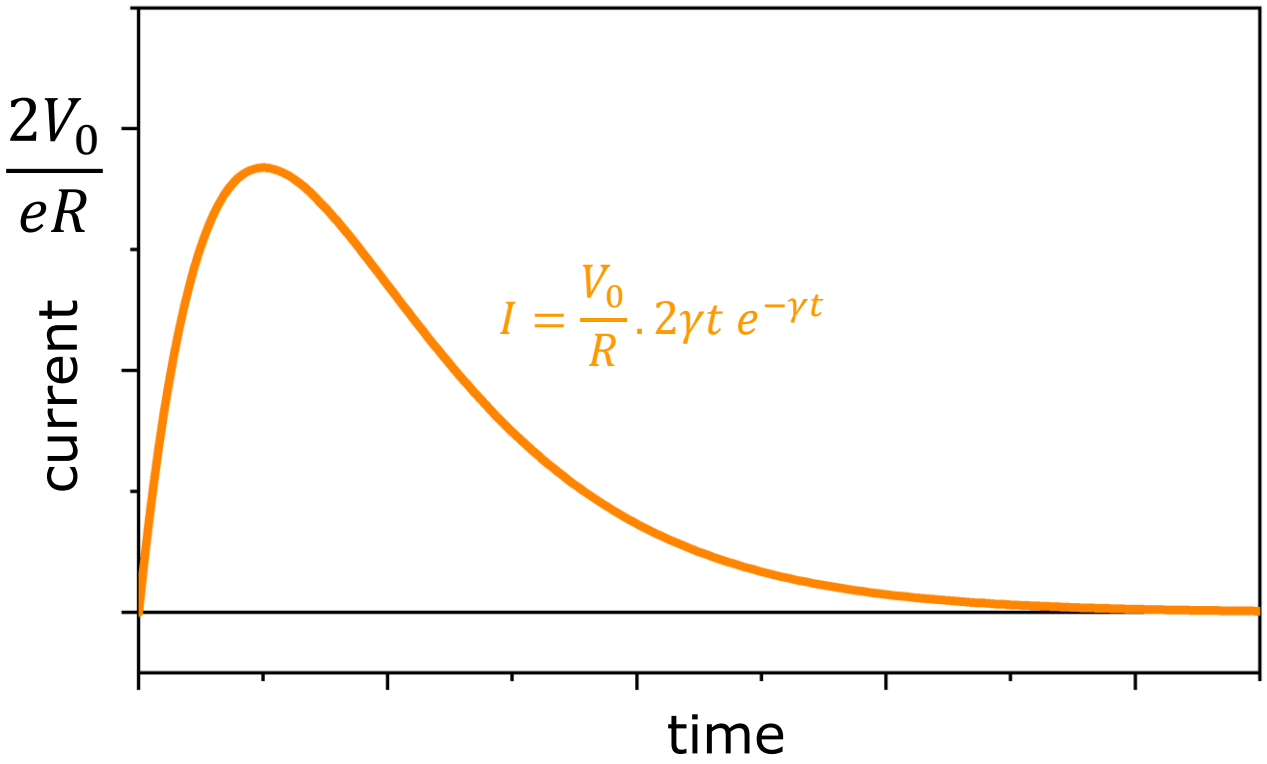
$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

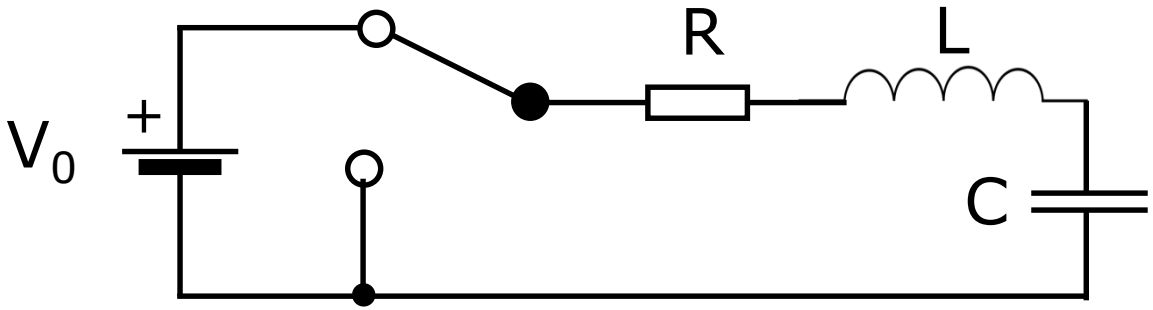
$$I = \frac{V_0}{R\sqrt{1-\omega_0^2/\gamma^2}}\left[\exp\left(-\left(1-\sqrt{1-\frac{\omega_0^2}{\gamma^2}}\right)\gamma t\right) - \exp\left(-\left(1+\sqrt{1-\frac{\omega_0^2}{\gamma^2}}\right)\gamma t\right)\right]$$

$$I \approx \frac{V_0}{R}\left[\exp\left(-\frac{t}{\tau_C}\right) - \exp\left(-\frac{t}{\tau_L}\right)\right] \quad (\gamma \gg \omega_0) \quad \tau_C = RC \quad \tau_L = \frac{L}{R}$$

Critically-damped $(\gamma = \omega_0)$

On blackboard

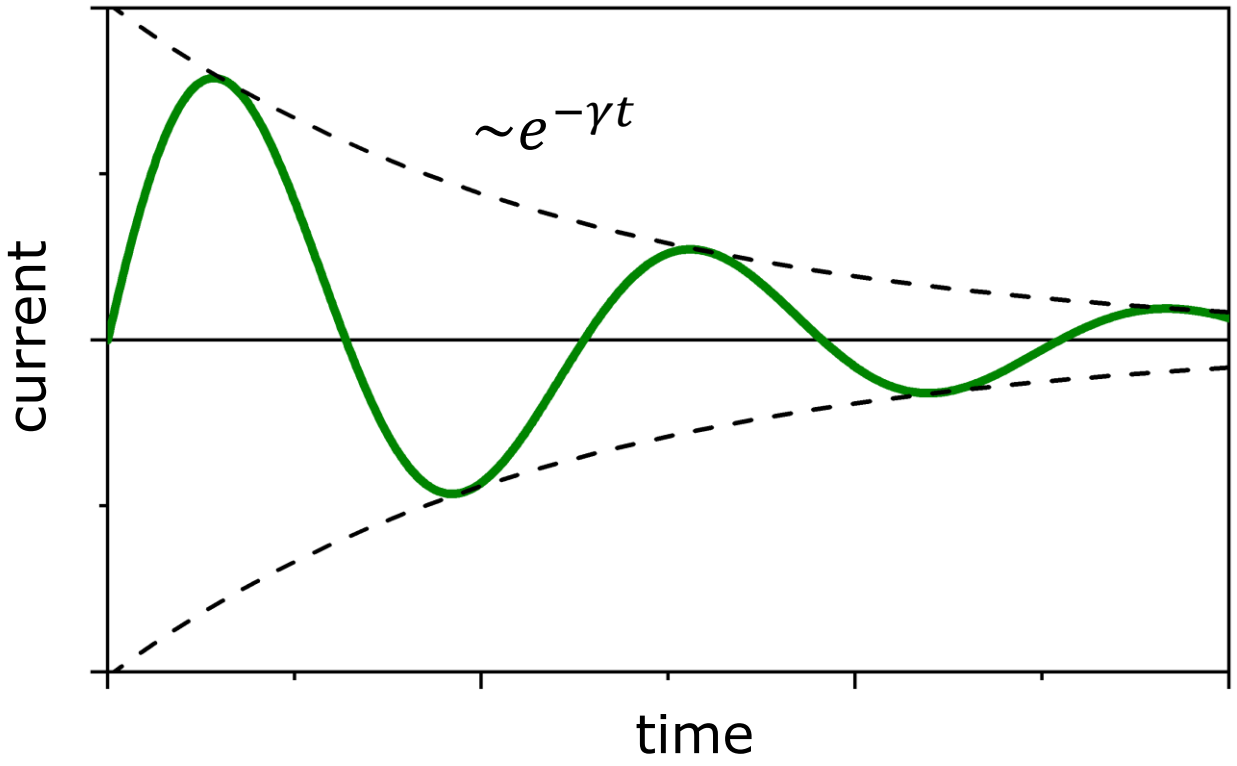


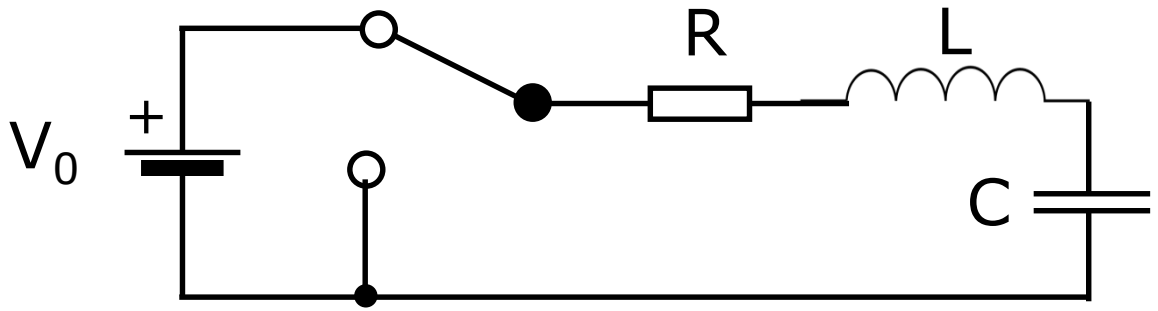


$\lambda_1 = \lambda_2 = -\gamma$

Under-damped ($\gamma < \omega_0$)

On blackboard





$$\lambda_{1,2} = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

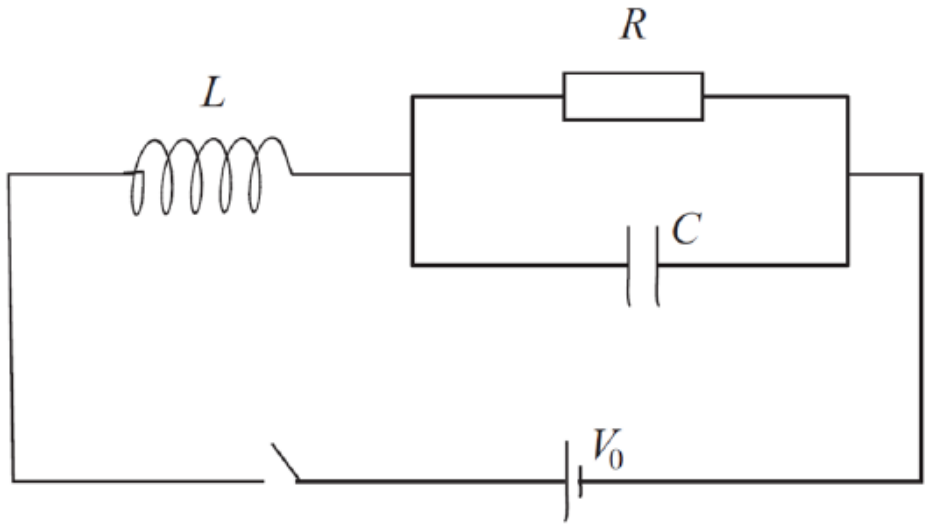
$$I = \frac{V_0}{R}\frac{2}{\sqrt{\omega_0^2/\gamma^2 - 1}} e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\ t\right)$$

Close to ω_0 but not exactly

$$I = V_0\sqrt{\frac{C}{L}}\frac{1}{\sqrt{1 - \gamma^2/\omega_0^2}} e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\ t\right)$$

Exam problem – to try yourself

b) The same network is connected to a battery, voltage V_0, as shown below. At time $t = 0$ the capacitor is uncharged and the switch is closed. By solving an appropriate differential equation, show that the current through the resistor is oscillatory provided $L < 4CR^2$. By considering the boundary conditions at $t = 0$ and as $t \to \infty$, sketch the form of this current as a function of time.



Application example

- Stereo system originally has 50W, 8Ω speakers

- You are now replacing the speakers from the options:
  - 50W, 4Ω
  - 50W, 16Ω

- Assuming the system and the original speakers were well matched which should you choose?



Circuit Theory
Lecture 5

Recap

- Capacitors store charge; they resist a change of voltage

$$V = \frac{Q}{C} \qquad W = \frac{1}{2}CV^2$$

- Inductors oppose a change in current

$$V = L\frac{dI}{dt} \qquad W = \frac{1}{2}LI^2$$

- Transient circuits

RC in series – decays to long-time values with $\tau = RC$

RL in series – decays to long-time values with $\tau = \frac{L}{R}$

LCR series – damped s.h.m. with $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = \frac{R}{2L}$

Over ($\gamma > \omega_0$), critical ($\gamma = \omega_0$) and under ($\gamma < \omega_0$) damped

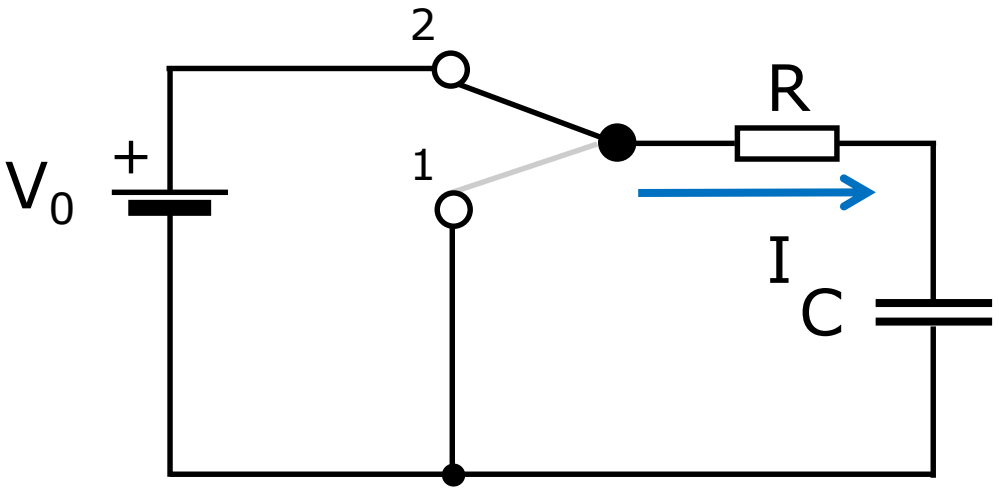
Today: AC circuit analysis

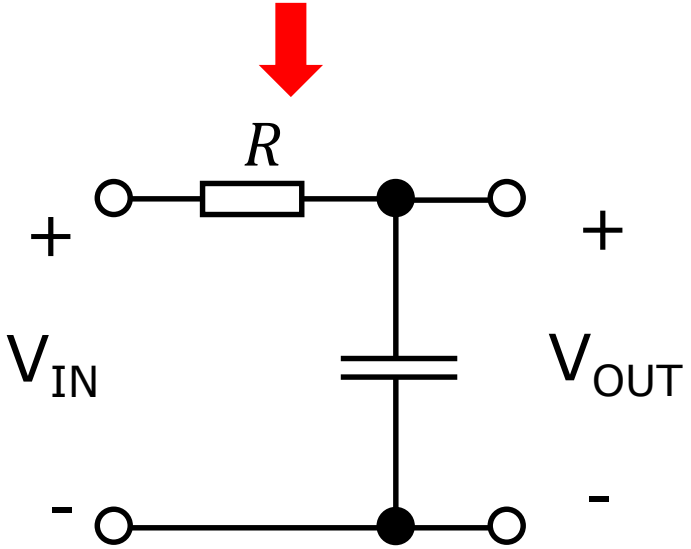
On blackboard

- LCR circuits - what if we don’t have switches but have some other input instead?
- Response of resistors, capacitors and inductors to AC sources
- Complex impedance

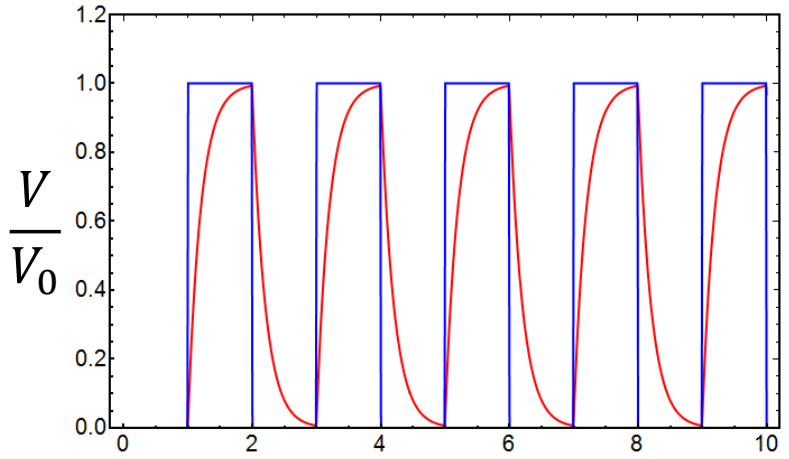
Square wave input

On blackboard

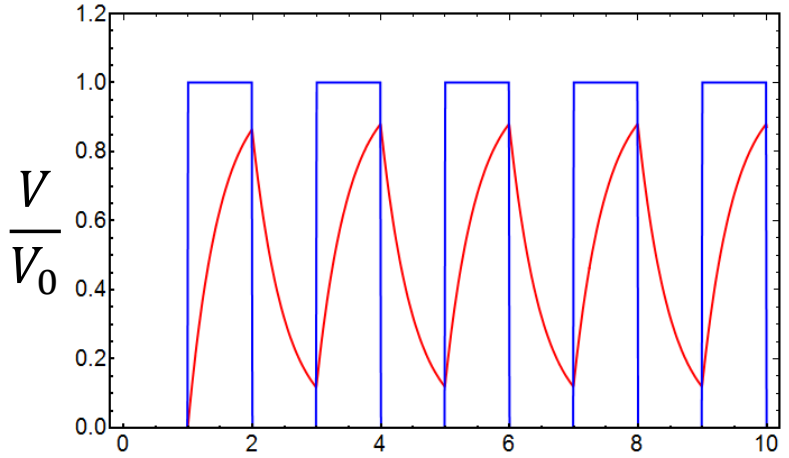




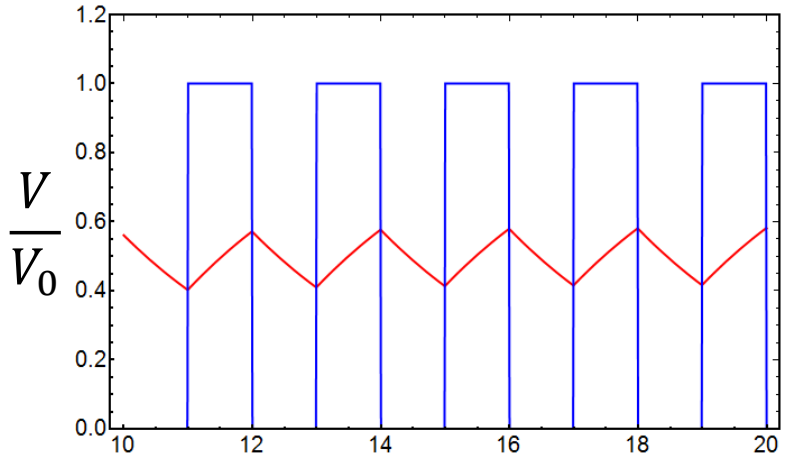
$T \gg \tau_c$



$T \approx \tau_c$

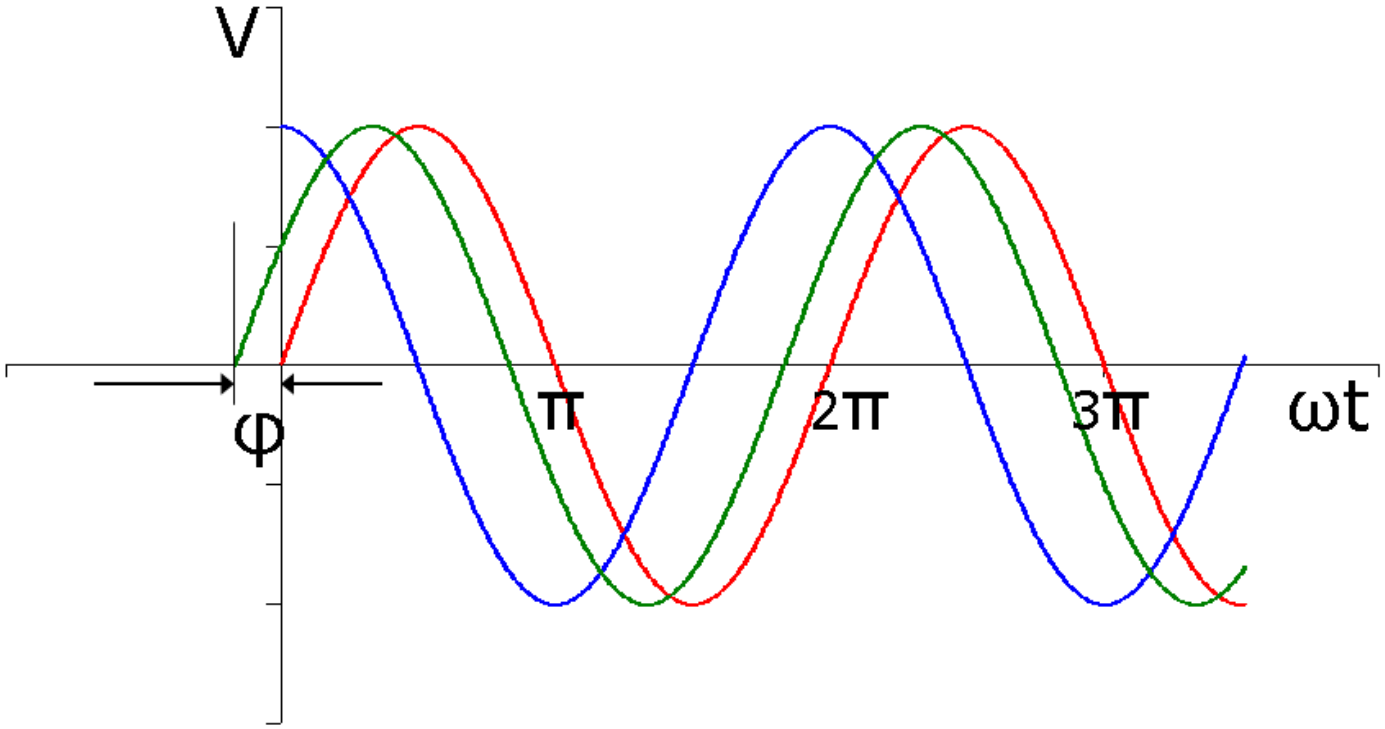


$T \ll \tau_c$



Sinusoidal voltages /currents

On blackboard



Amplitude

Phase

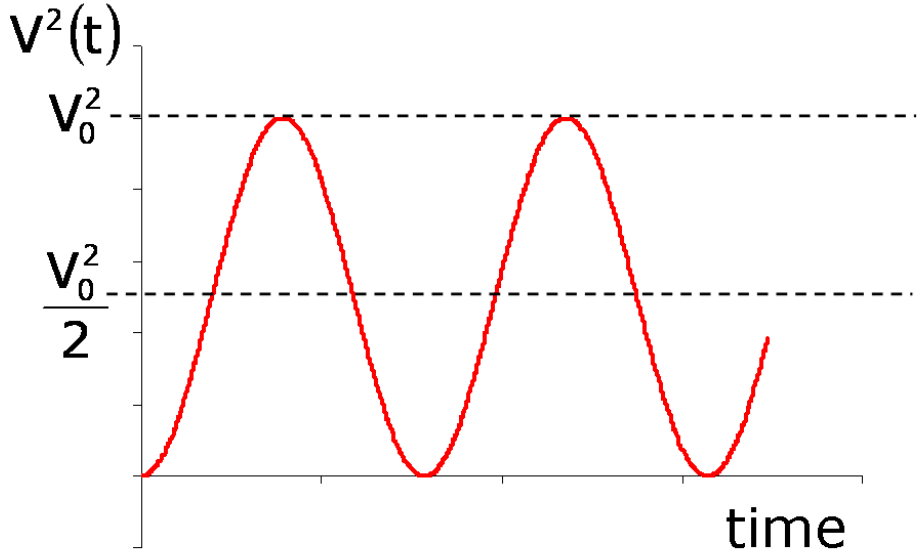
$$V = V_0 \sin(\omega t)$$
$$V = V_0 \cos(\omega t)$$
$$V = V_0 \sin(\omega t + \phi)$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Power and RMS values

For resistive load $\langle P \rangle = \langle IV \rangle = \left\langle \frac{V^2}{R} \right\rangle = \frac{V_{\text{rms}}^2}{R}$

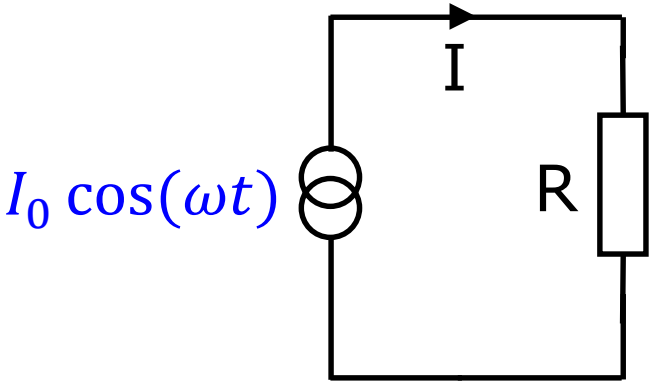
$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T V^2(t)\,dt}$$



For sinusoid $V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$

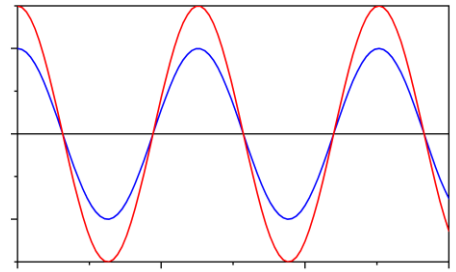
Apply $I = I_0 \cos(\omega t)$ to R, C, L

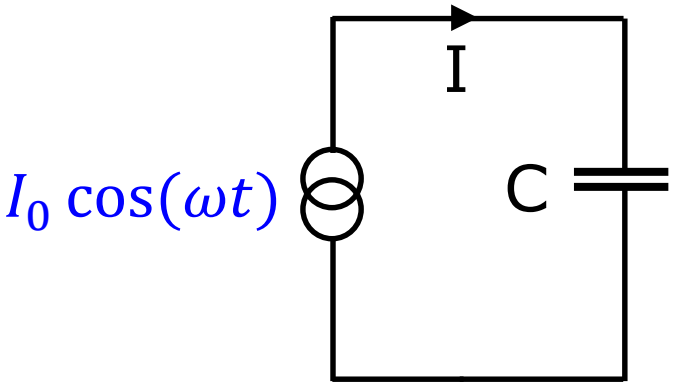
On blackboard



$$V_R = RI = RI_0 \cos(\omega t)$$

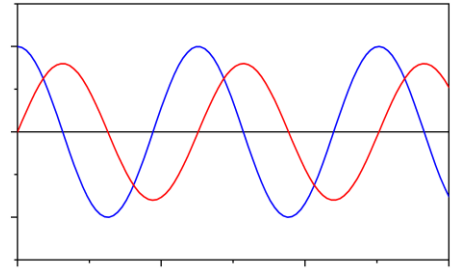
Voltage in phase with current

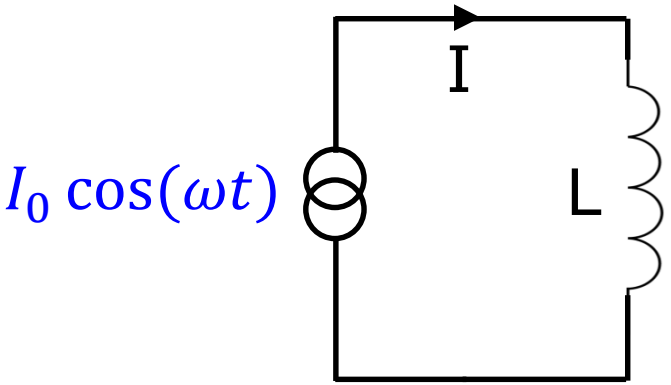




$$V_c = \frac{Q}{C} = \frac{1}{C}\int I_0 \cos(\omega t)dt = \frac{1}{\omega C} I_0 \sin(\omega t) = \frac{1}{\omega C} I_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

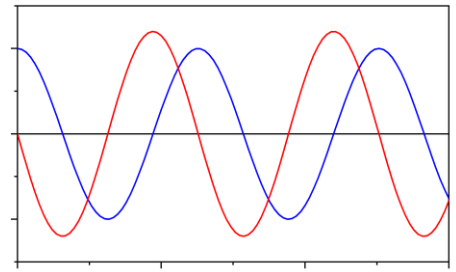
Voltage lags current by $\pi/2$





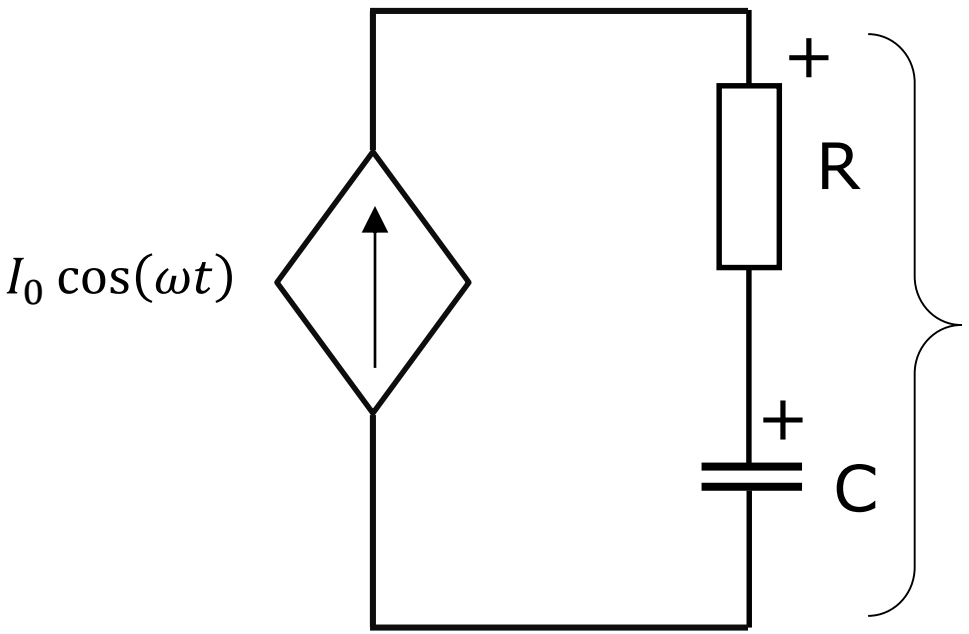
$$V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t) = \omega L\, I_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

Voltage leads current by $\pi/2$



Adding voltages

On blackboard



$$V = V_R + V_c$$

voltages out of phase…

$$V = I_0 \left(R \cos(\omega t) + \frac{1}{\omega C} \sin(\omega t) \right)$$

$$V = I_0 A \cos(\omega t + \phi)$$

$$A = \sqrt{R^2 + \left(\frac{1}{\omega C} \right)^2} \qquad \phi = -\arctan\left(\frac{1}{\omega RC} \right)$$

Complex Impedance

On blackboard

Tells you both the amplitude and phase of the voltage response when applying an oscillating current

$$Z = Ae^{i\phi} = |Z|e^{i\phi}$$

Resistor $Z_R = R$

Capacitor $Z_C = \frac{1}{\omega C} e^{-\frac{i\pi}{2}} = \frac{1}{i\omega C}$

Inductor $Z_L = \omega L e^{\frac{i\pi}{2}} = i\omega L$

$$Z = R + iX$$

Resistance Reactance

Generalised Ohm's law $\tilde{V} = \tilde{I}Z$

On blackboard

$$I = I_0 \cos(\omega t) = \mathrm{Re}\left(I_0 e^{i\omega t}\right) = \mathrm{Re}(\tilde{I})$$

$$V = I_0|Z| \cos(\omega t + \phi) = \mathrm{Re}\left(I_0|Z|e^{i(\omega t+\phi)}\right)$$

$$V_0 \cos(\omega t + \phi) = \mathrm{Re}\left(I_0|Z|e^{i\phi}e^{i\omega t}\right)$$

$$\mathrm{Re}\left(V_0 e^{i(\omega t+\phi)}\right) = \mathrm{Re}\left(I_0 Z e^{i\omega t}\right)$$

$$\mathrm{Re}\left(\tilde{V}\right) = \mathrm{Re}(\tilde{I}Z)$$

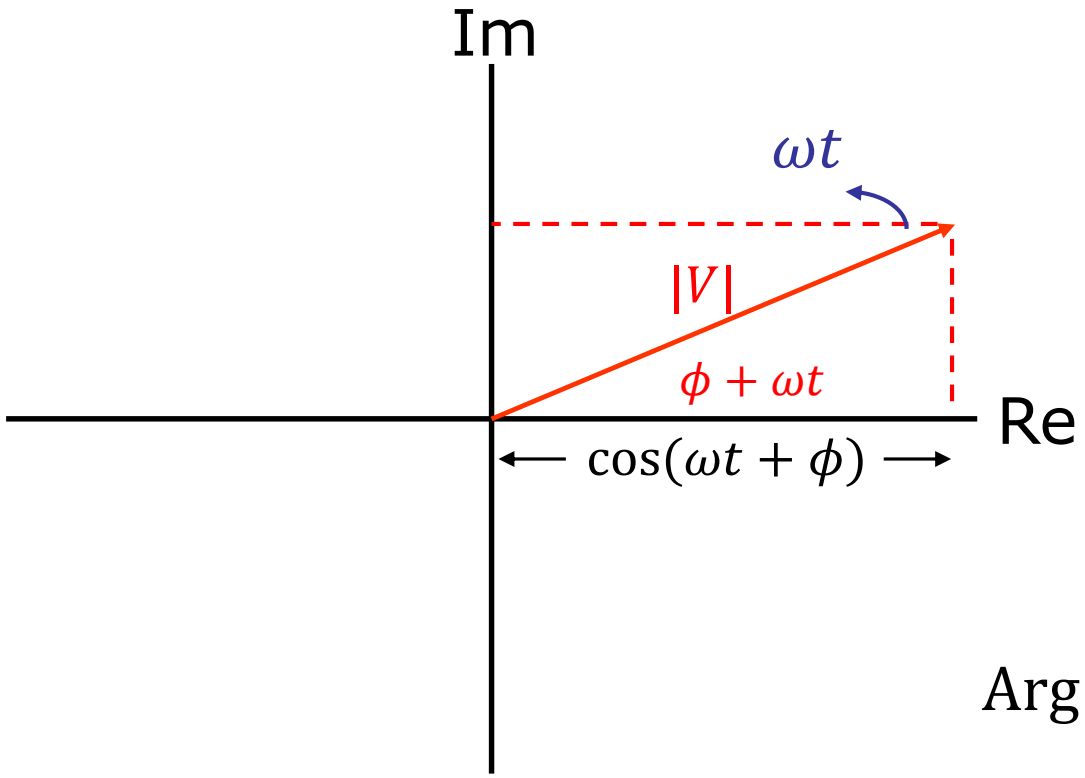
$$\tilde{V} = \tilde{I}Z$$

$\tilde{V}$ and $\tilde{I}$ are phasors

Phasors

On blackboard

- Rotate around in complex plane
- Actual voltage is projection onto real axis



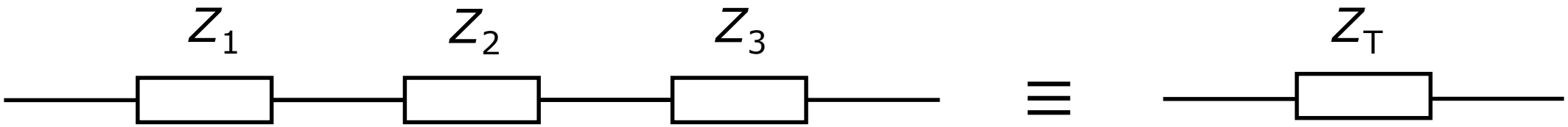
$$\tilde{V} = \tilde{I}Z$$

$$\mathrm{Arg}(\tilde{V}) = \mathrm{Arg}(Z) + \mathrm{Arg}(\tilde{I})$$

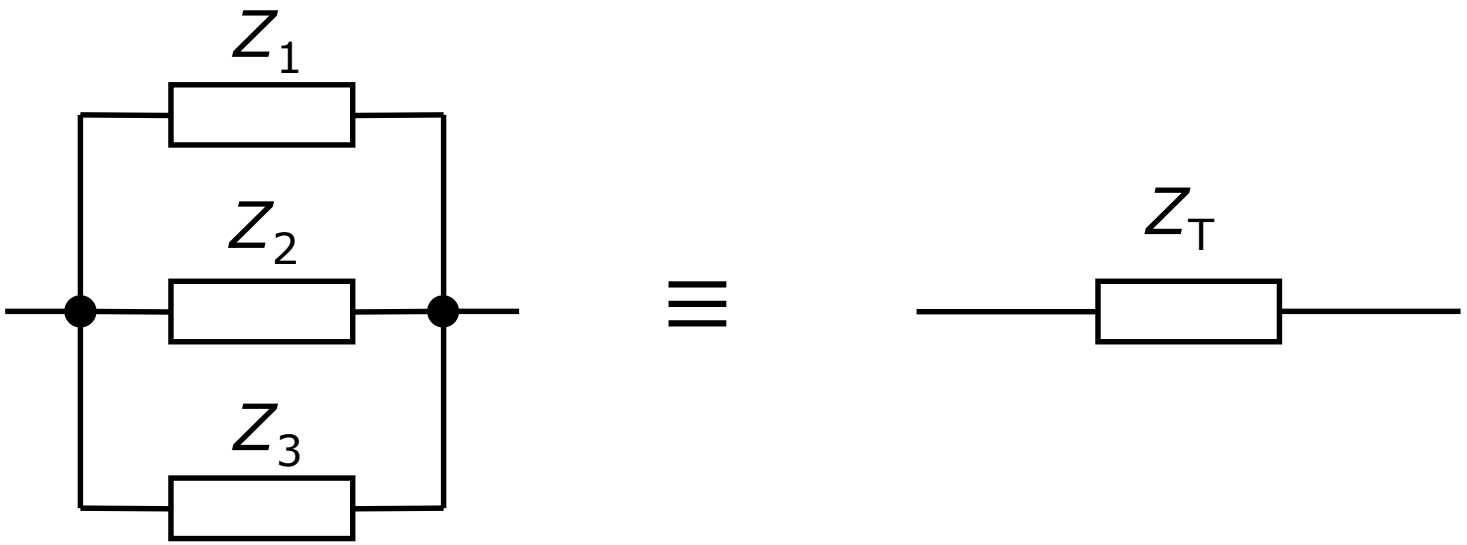
$$|\tilde{V}| = |Z||\tilde{I}|$$

Series and parallel impedances

On blackboard



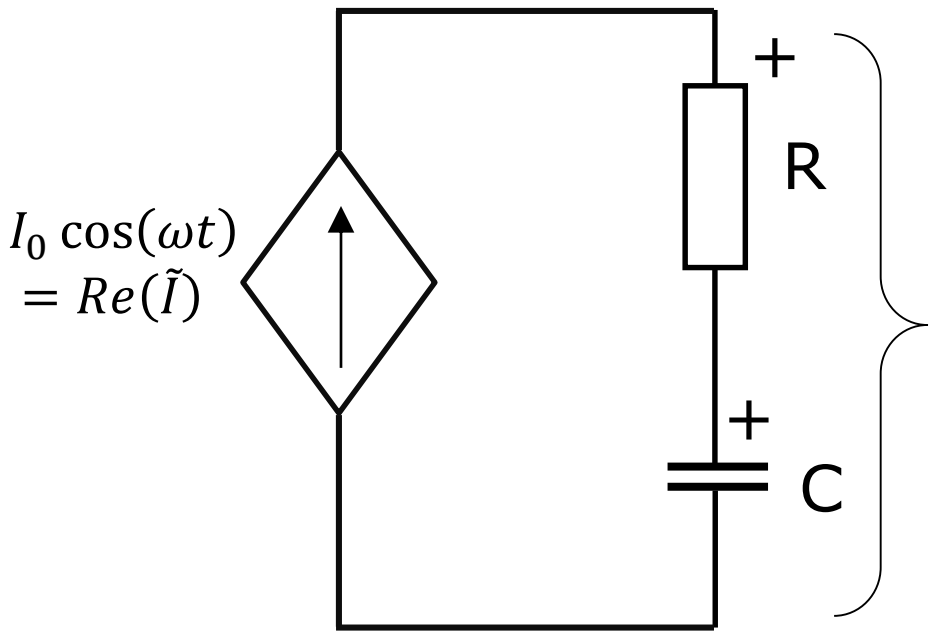
$$Z_T = \sum_n Z_n$$



$$\frac{1}{Z_T} = \sum_n \frac{1}{Z_n}$$

Adding voltages revisited

On blackboard



$$V = V_R + V_c = Re(\tilde{V})$$

$$\tilde{V} = \tilde{I}Z$$

$$Z = R + \frac{1}{i\omega C}$$

$$V = I_0 |Z| \cos(\omega t + \phi)$$

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \qquad \phi = -\arctan\left(\frac{1}{\omega RC}\right)$$

Circuit Theory

Lecture 6

Recap

- Response to AC current
  - Resistors $V = I_0 R \cos(\omega t)$
  - Capacitors $V = \frac{I_0}{\omega C} \cos\left(\omega t - \frac{\pi}{2}\right)$ Voltage lags current
  - Inductors $V = I_0 \omega L \cos\left(\omega t + \frac{\pi}{2}\right)$ Voltage leads current
- Complex impedance (Z)
  - Describes amplitude and phase of *V* response to AC I
  - $Z = |Z| e^{i\phi} = R + iX$
  - Generalised Ohm's law: $\tilde{V} = \tilde{I} Z$ (or often $V = IZ$)
  - Combining in series $Z_T = \sum_n Z_n$ and parallel $\frac{1}{Z_T} = \sum_n \frac{1}{Z_n}$

$$Z_R = R \qquad Z_C = \frac{1}{i\omega C} \qquad Z_L = i\omega L$$

Today: some examples

- Series and parallel LCR circuits

- Filters
  - RC low pass filter
  - RL high pass filter
  - Band pass filter

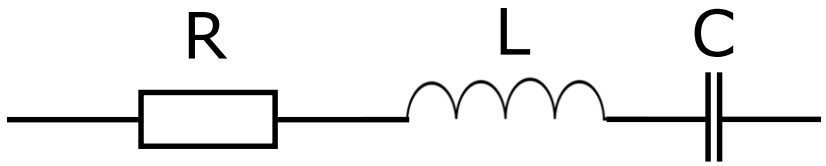
Bode plots

Limits of Z_c and Z_L

On blackboard

- **Capacitor**
  - At low frequencies (like DC) → **Open circuit**
    - Not a surprise, it's got a gap!
  - At high frequencies ("fast") → **Short circuit!**
- **Inductor**
  - At low frequencies → **Short circuit**
    - Not a surprise, it's just a wire really
  - At high frequencies → **Open circuit!**
- **Sometimes this can help you with your intuition on the circuit's behaviour.**

LCR series circuit



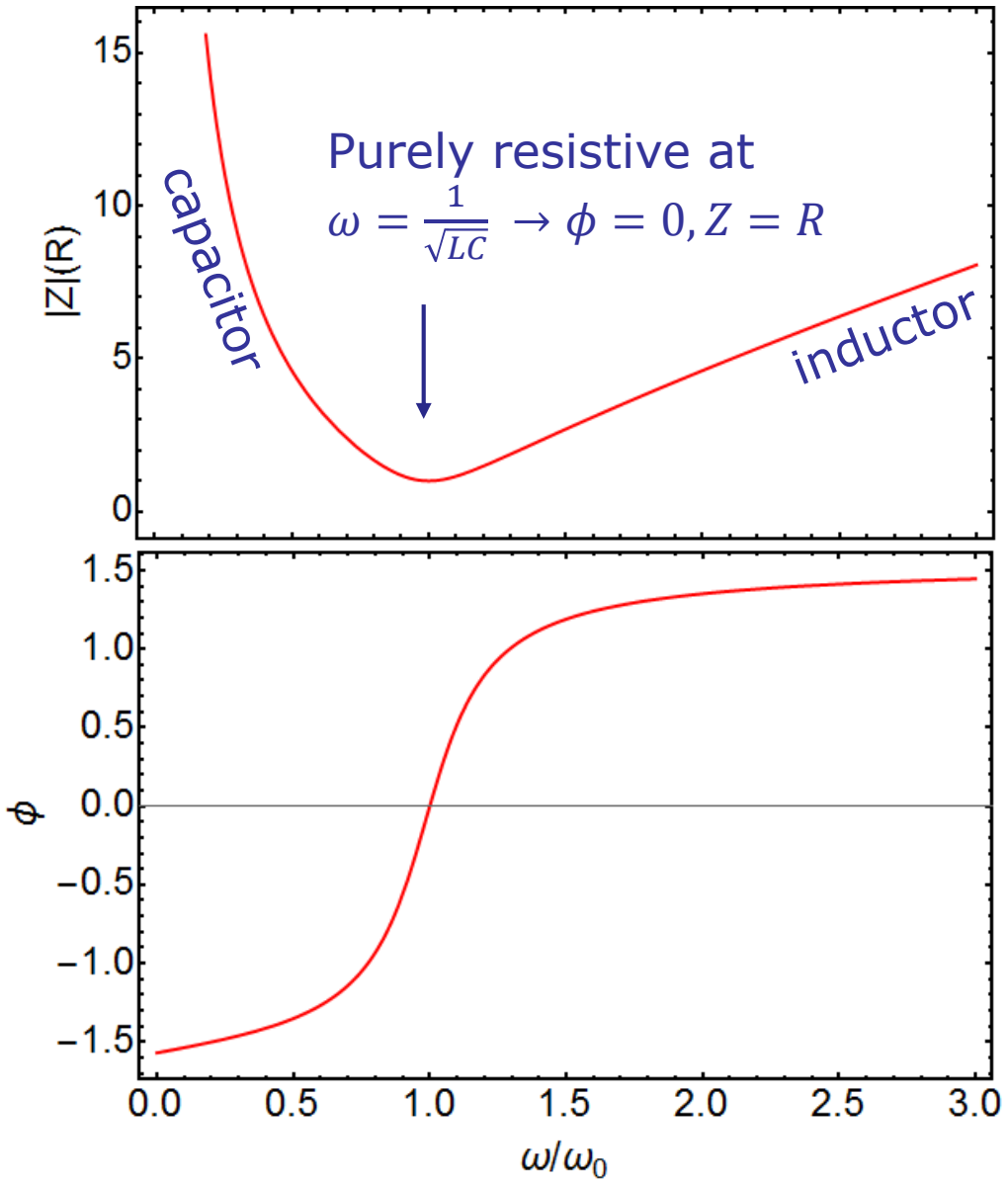
On blackboard

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

$$\frac{Z}{R} = 1 + i\frac{\omega_0}{2\gamma}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

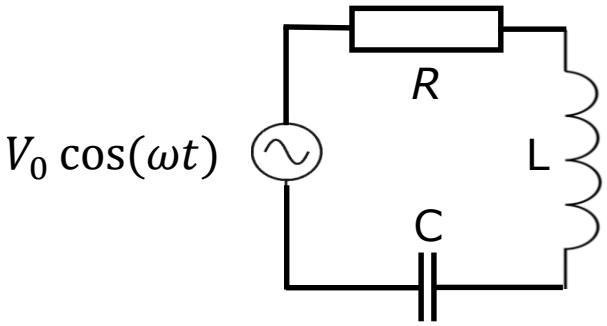
$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$



LCR series circuit – voltage driven

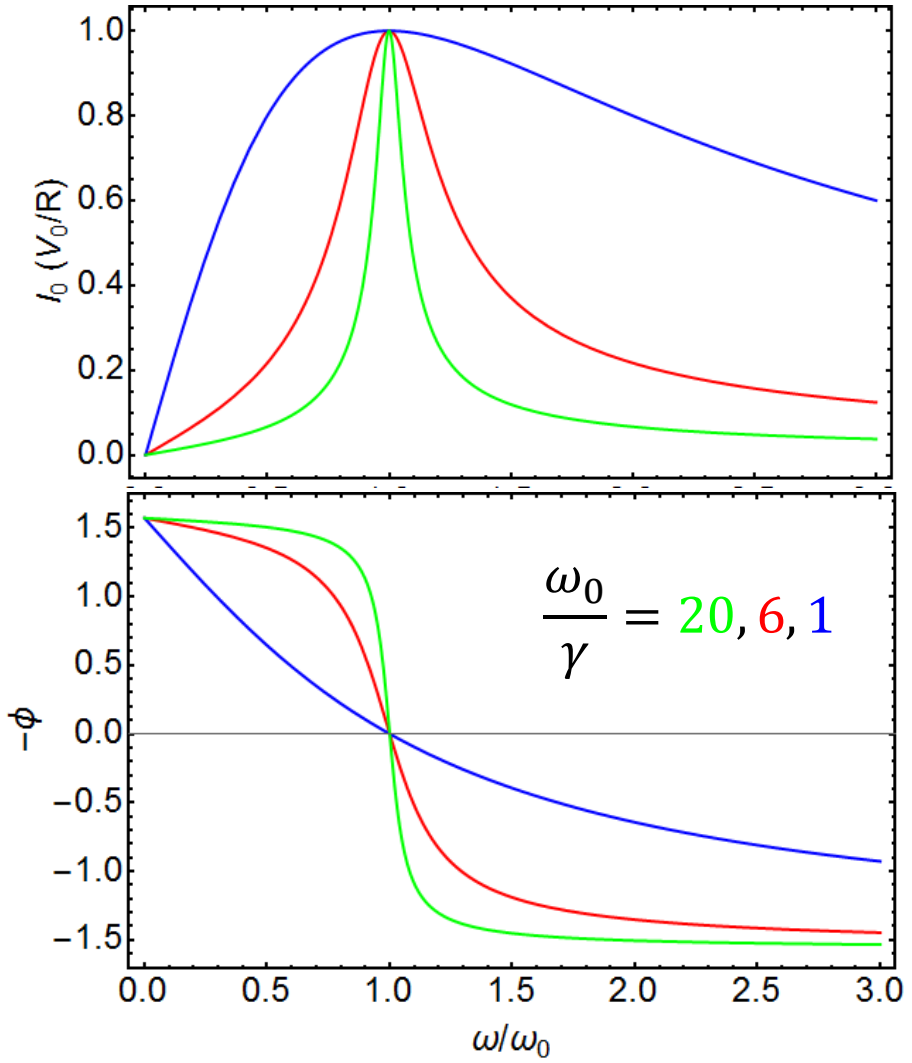
On blackboard



$$\frac{Z}{R} = 1 + i\frac{\omega_0}{2\gamma}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

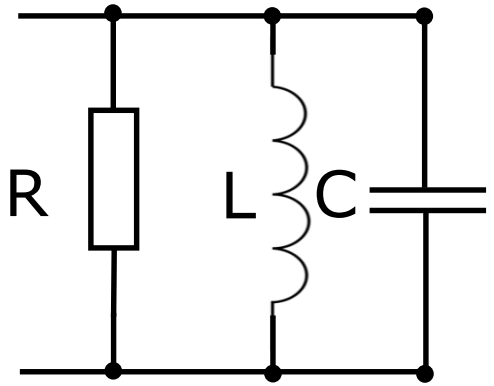
$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{\tilde{V}}{|Z|}e^{-i\phi}$$

$$I = I_0 \cos(\omega t + (-\phi))$$



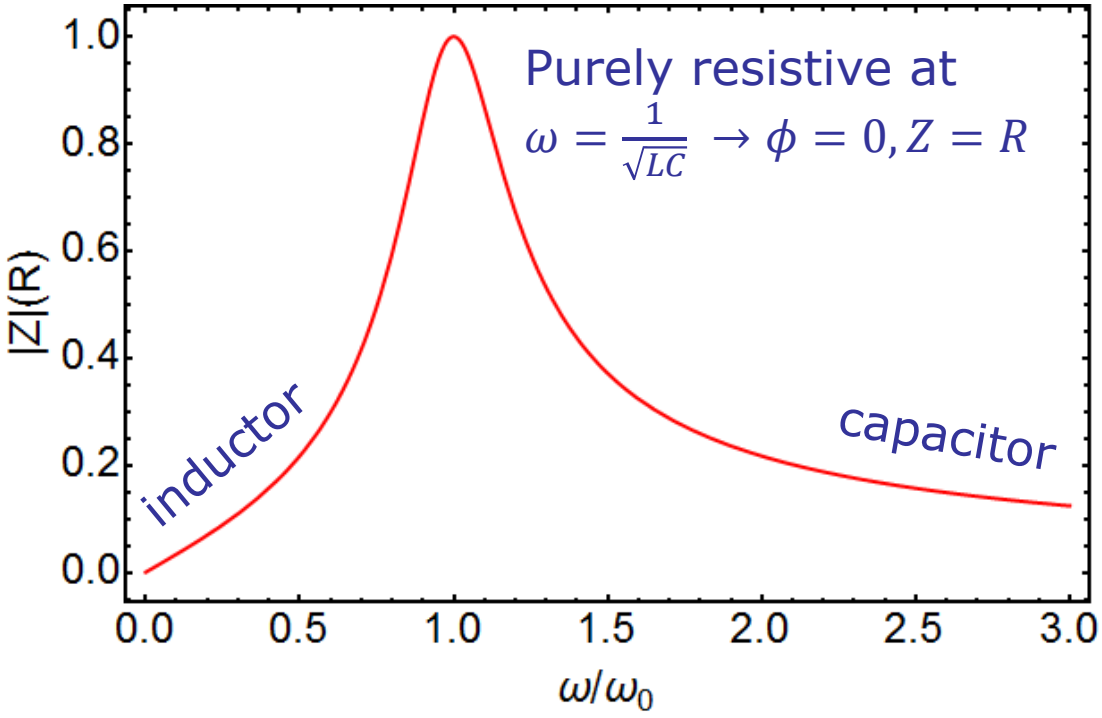
LCR parallel circuit

On blackboard



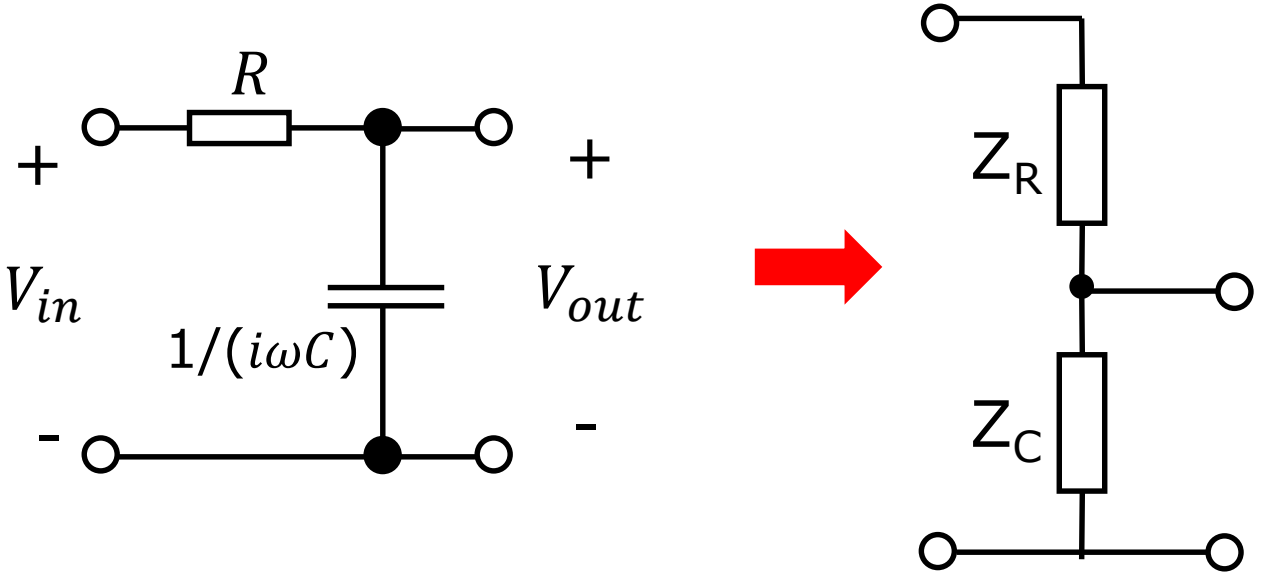
$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C$$

$$\frac{1}{Z} = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)$$



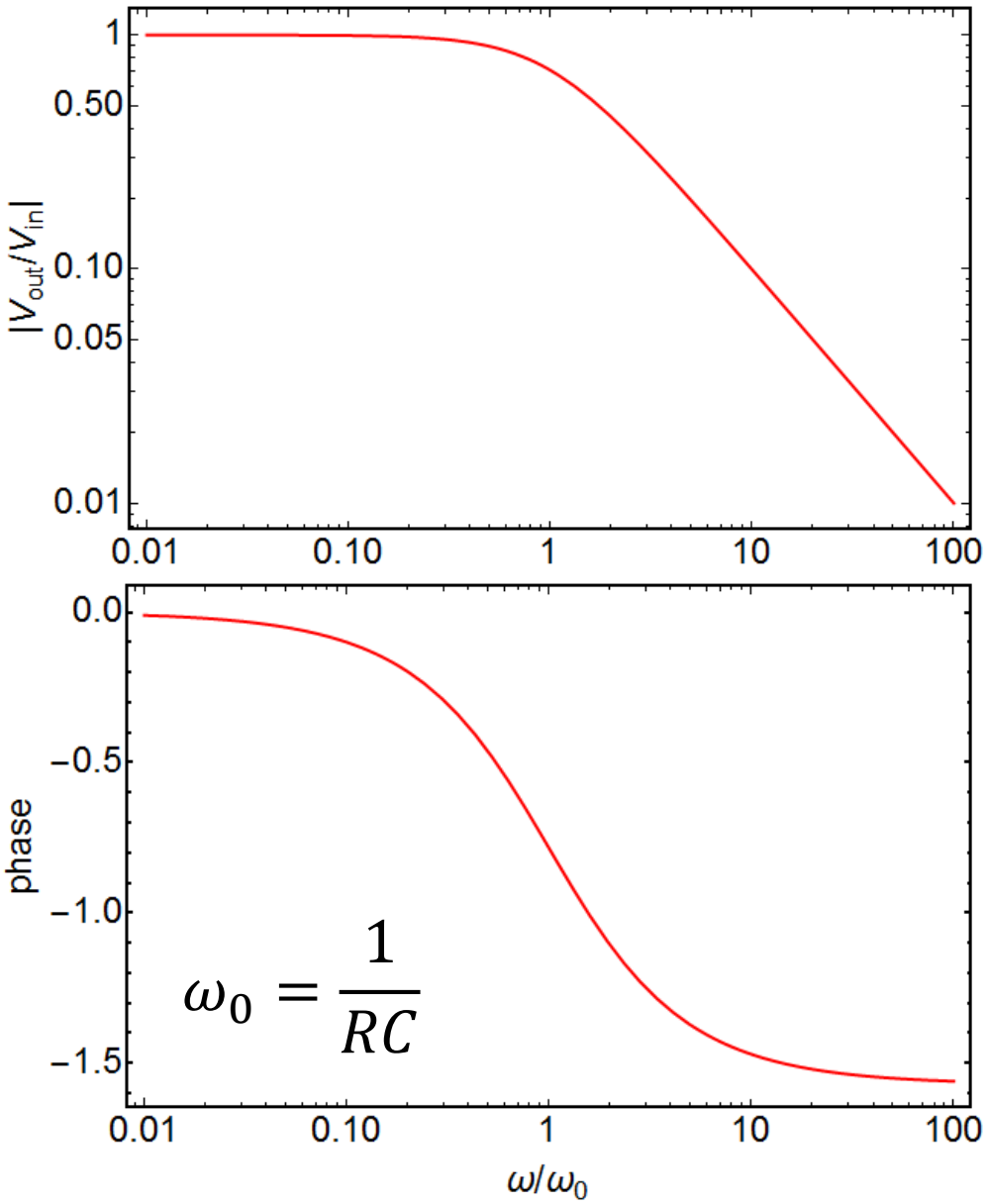
RC low pass filter

On blackboard



$$\frac{V_{out}}{V_{in}} = \frac{Z_c}{Z_R + Z_C} = \frac{1}{1 + i\omega RC} = \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{i\phi}$$

$$\phi = -\arctan(\omega RC)$$



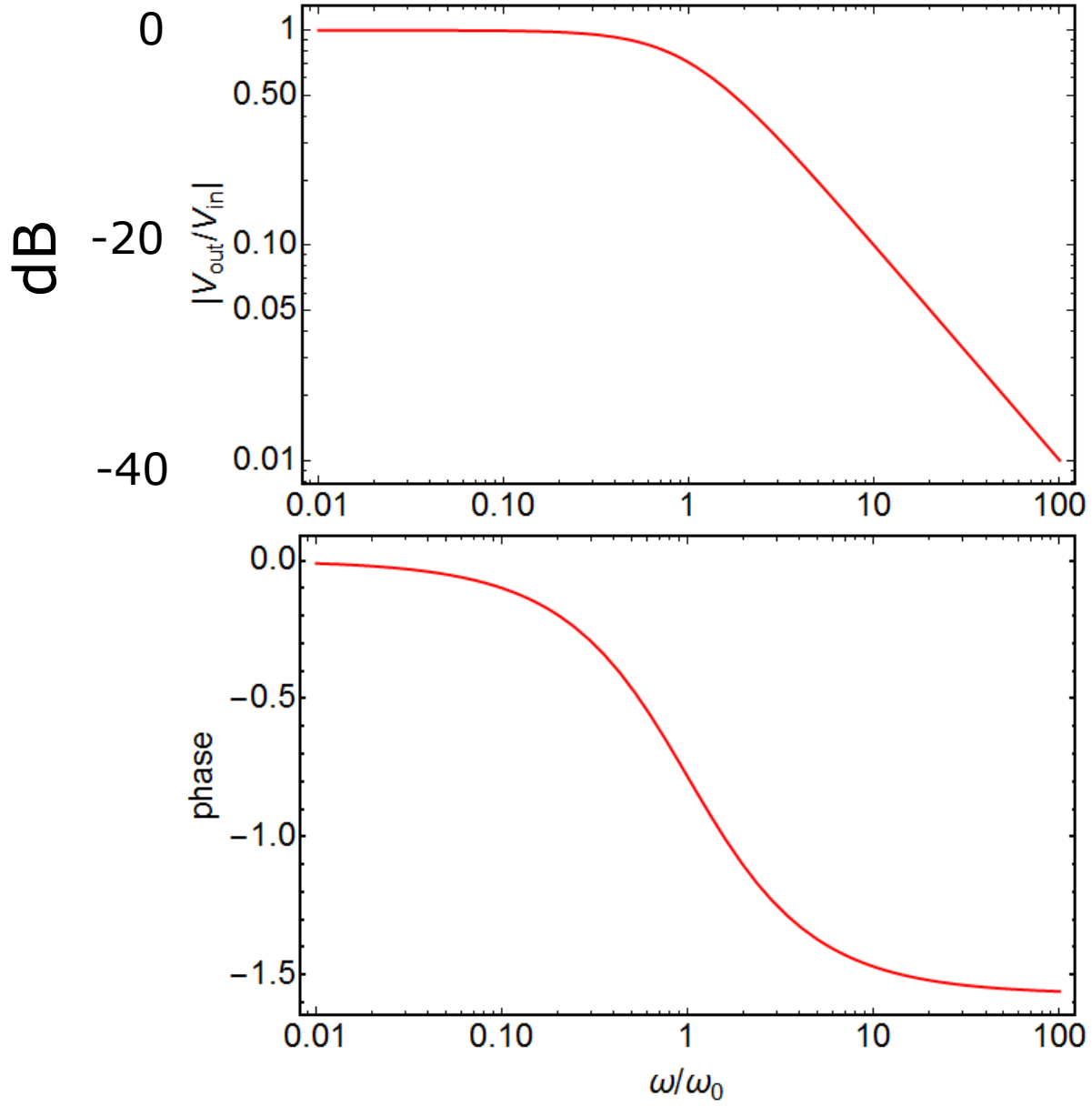
Bode plots and Decibels

On blackboard

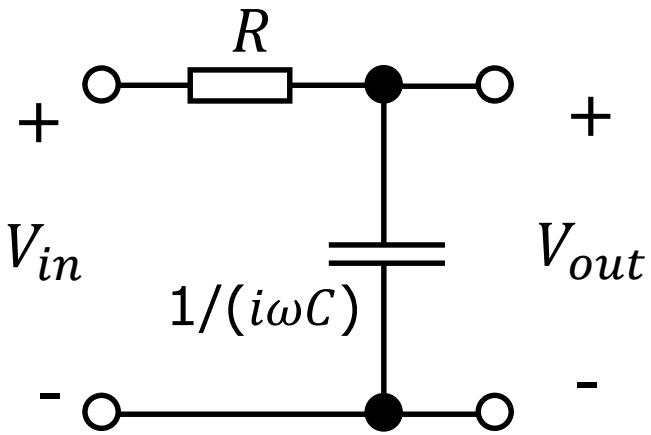
Logarithm of power ratio:

$V_{\text{out}}/V_{\text{in}}$	dB
10	20
1	0
0.1	-20
0.01	-40
0.001	-60

$$\text{dB} = 10\log_{10}\left|\frac{V_{out}^2}{V_{in}^2}\right| = 20\log_{10}\left|\frac{V_{out}}{V_{in}}\right|$$

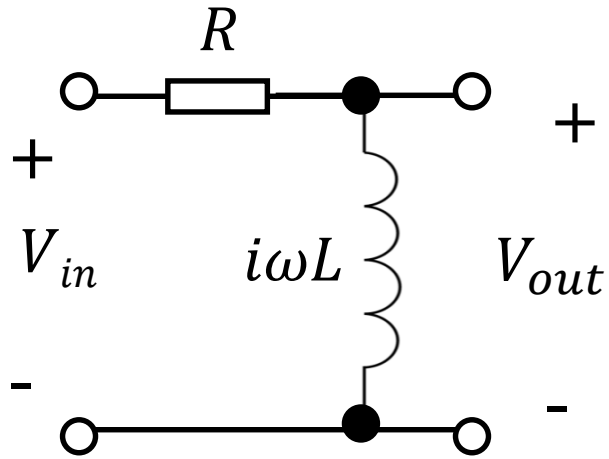


Bode plots (amplitude + phase)



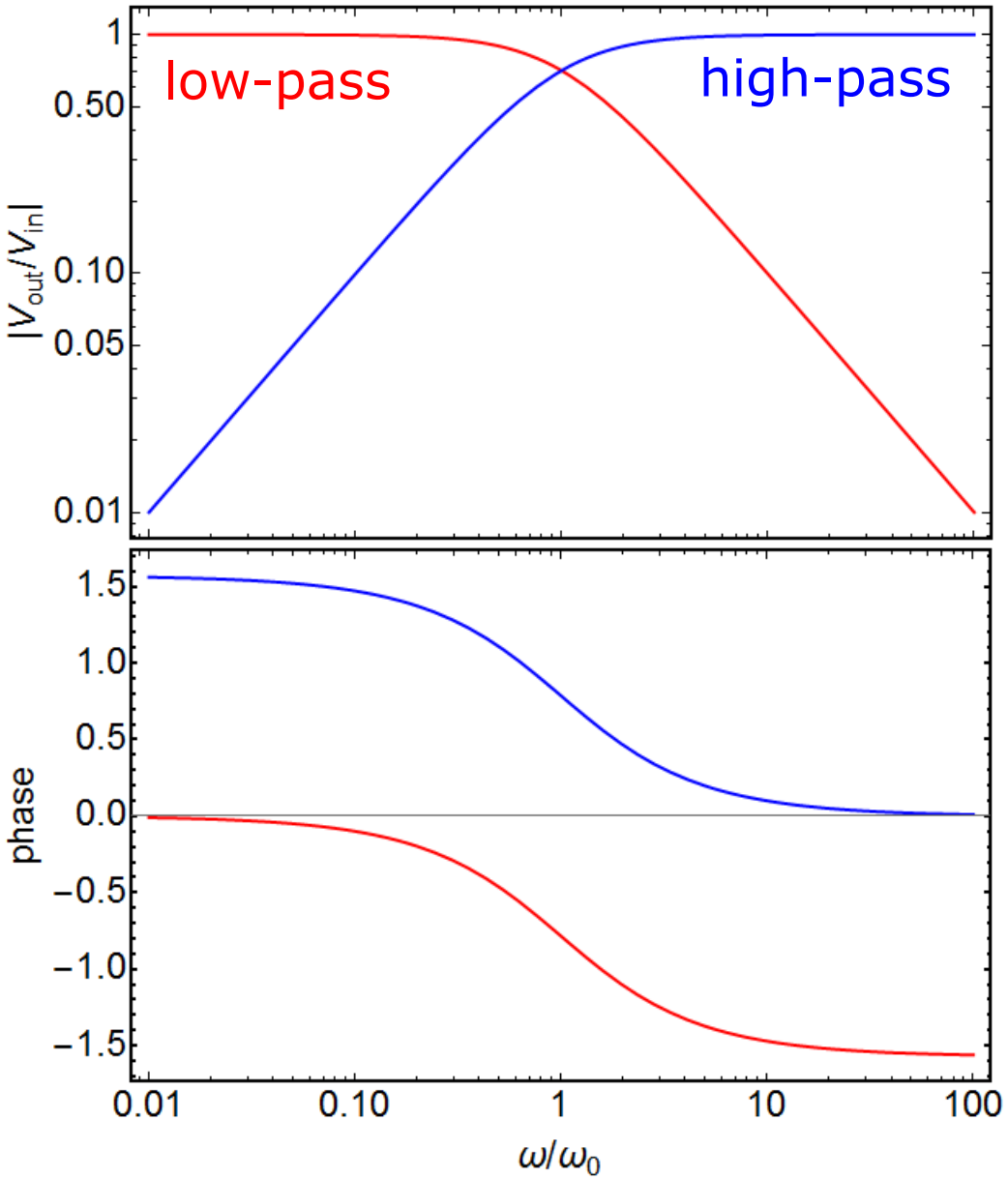
Low pass

$$\omega_0 = \frac{1}{RC}$$

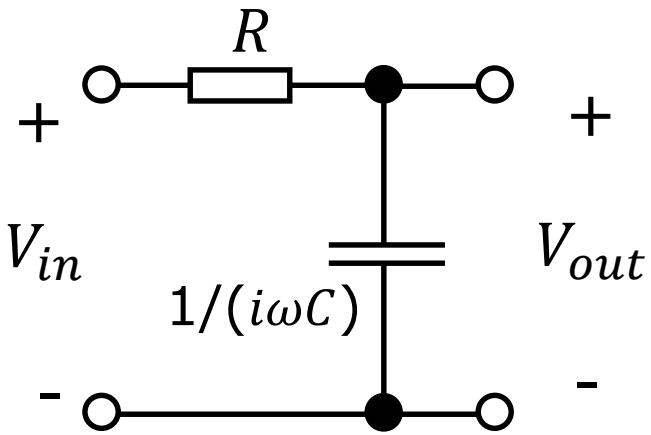


High pass

$$\omega_0 = \frac{R}{L}$$

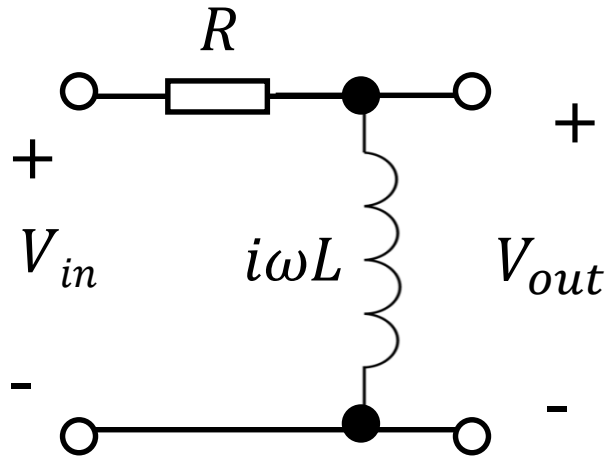


Example



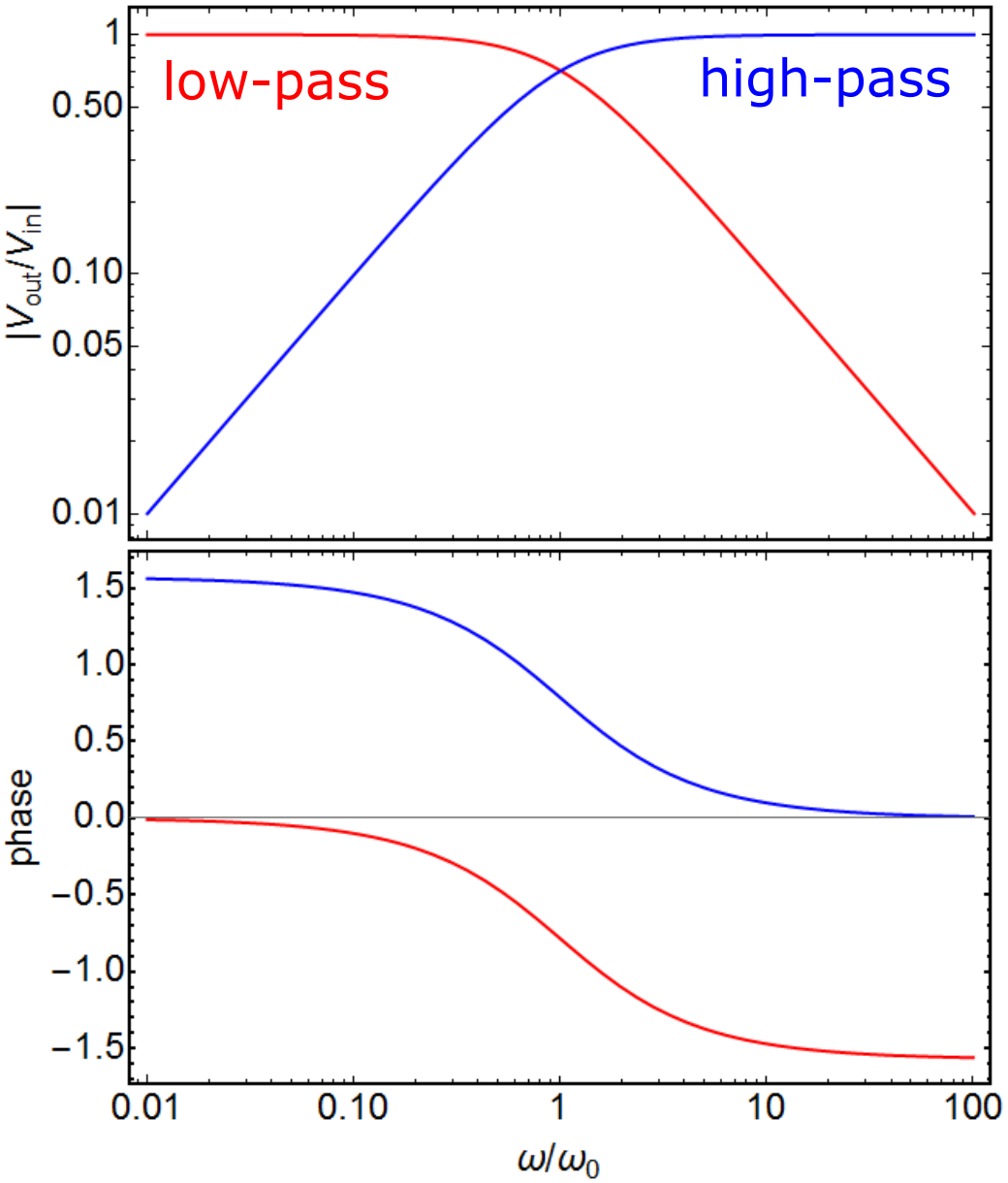
Low pass

$$\omega_0 = \frac{1}{RC}$$



High pass

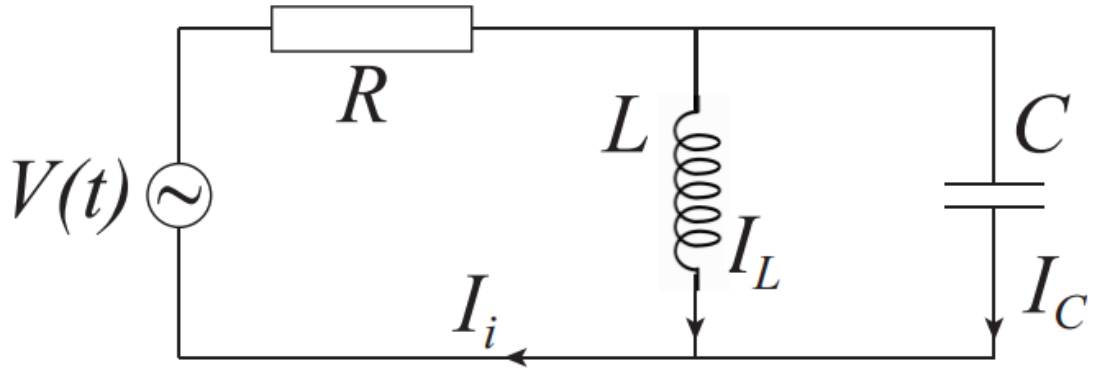
$$\omega_0 = \frac{R}{L}$$



CP2 2012

On blackboard

7. The diagram below shows a parallel LC resonance circuit in series with a resistance R, driven by a voltage source $V(t) = V_i \sin \omega t$.



(i) By replacing the voltage source and resistor combination with a Norton equivalent, or otherwise, show that the complex amplitude of the current in the inductor can be written as

$$I_L = \frac{1}{1 + j\omega\frac{L}{R} - \omega^2 LC} \times \frac{V_i}{R} \ ,$$

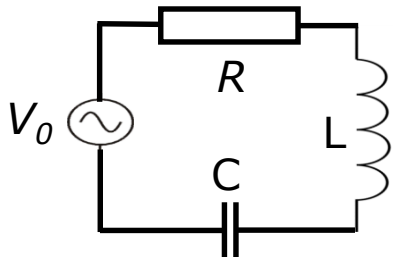
and find a similar expression for the complex amplitude of the current in the capacitor.

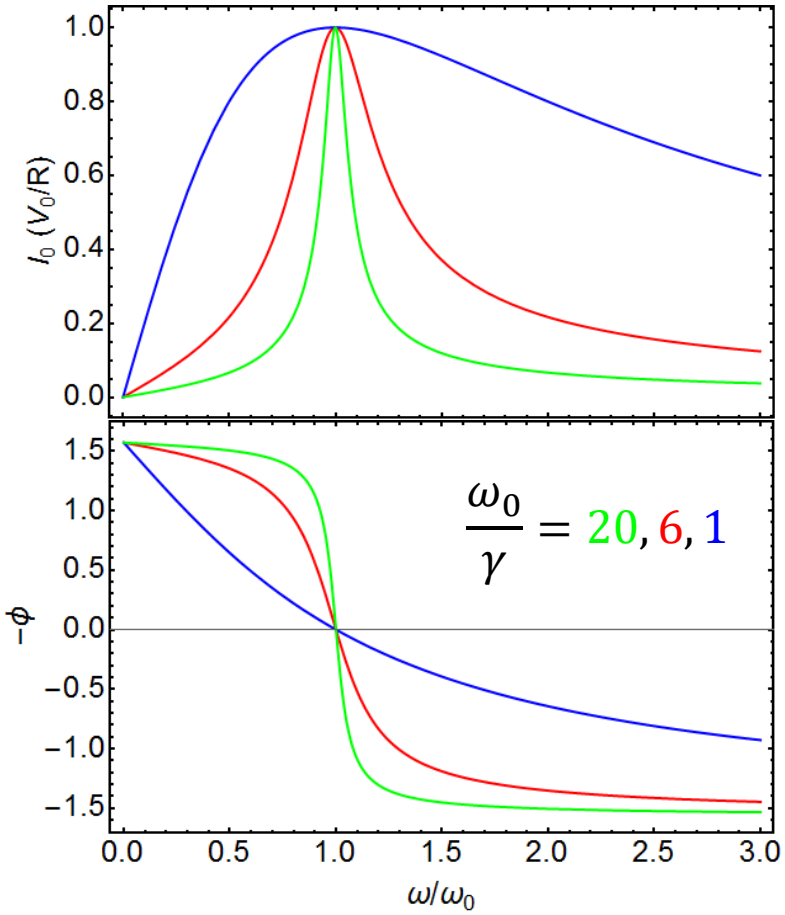
[8]

Circuit Theory
Lecture 7

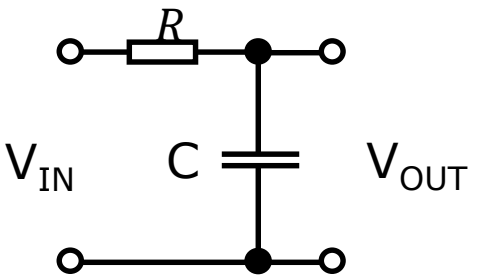
Recap

Driven circuits

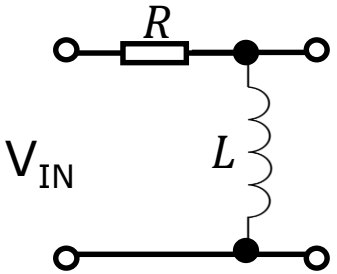




Filters

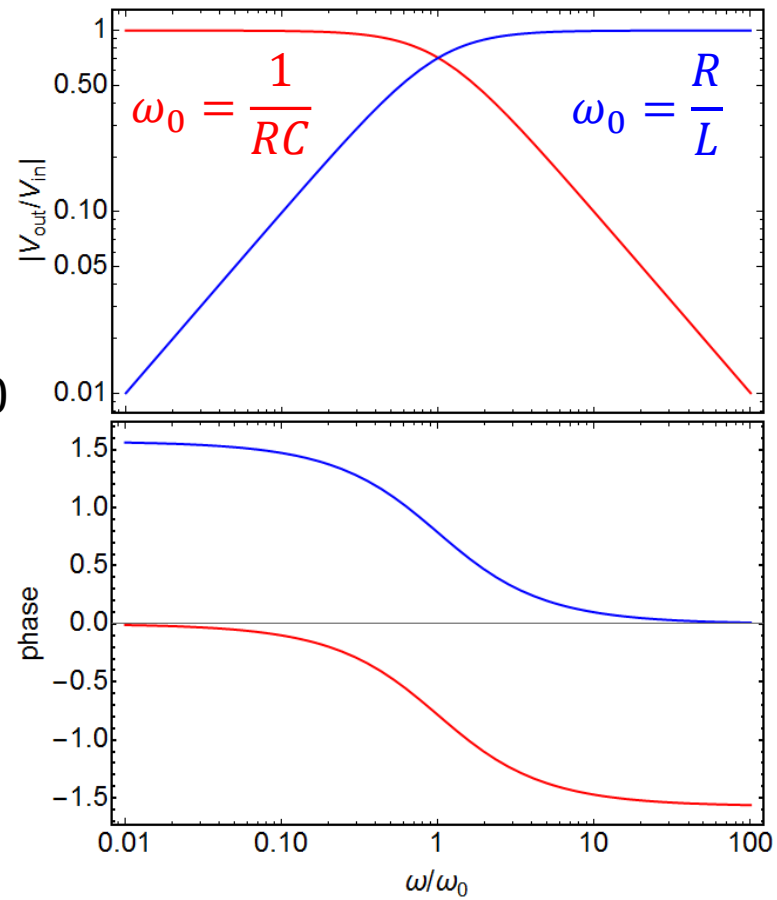


low-pass



high-pass

$$\text{dB} = 20 \log_{10} \left| \frac{V_{out}}{V_{in}} \right|$$



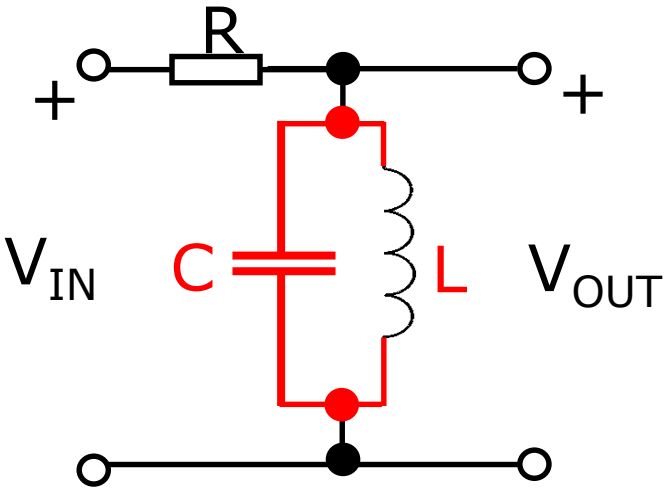
Vevox question

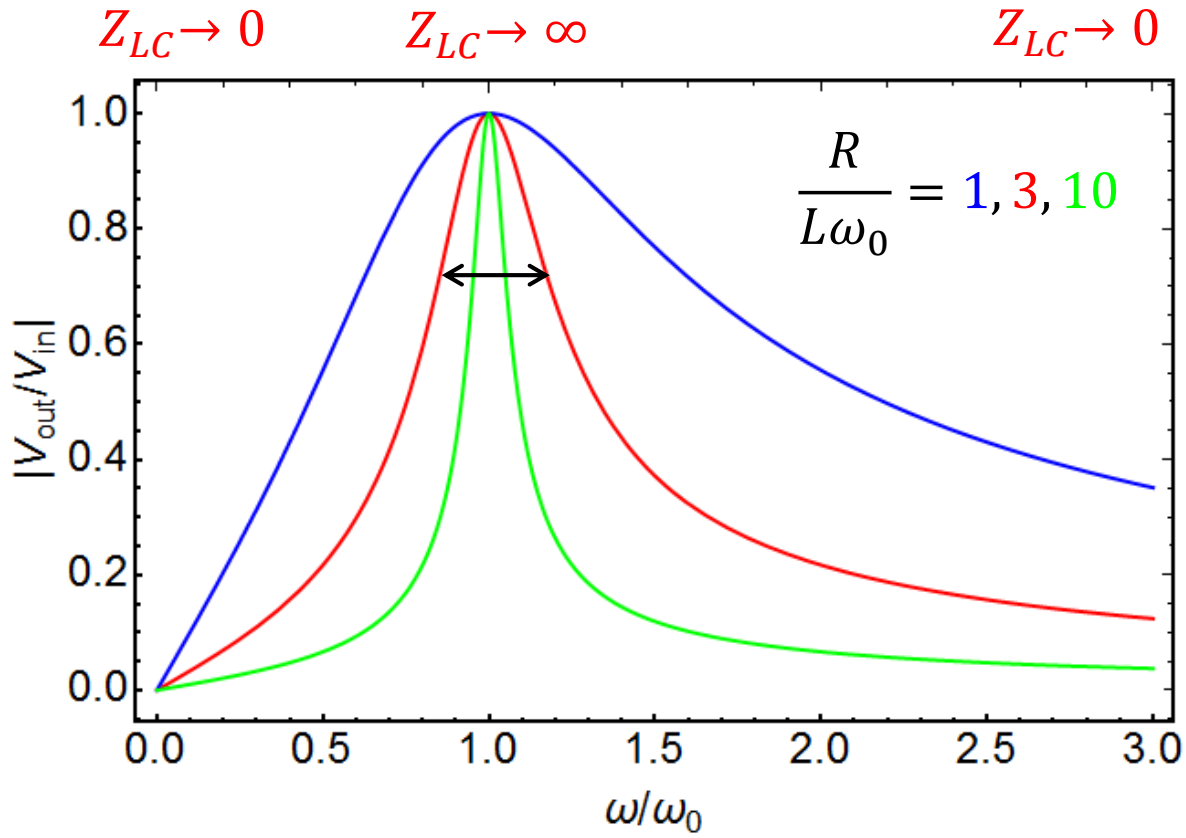
Today: AC circuit analysis

- Band pass filter
- Bridge circuits
- Power dissipation in AC circuits

Bandpass filter

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{iR}{L\omega_0}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$





Here larger R gives sharper feature!!

$$\Delta\omega_{FWHM} = \frac{L\omega_0^2}{R}$$

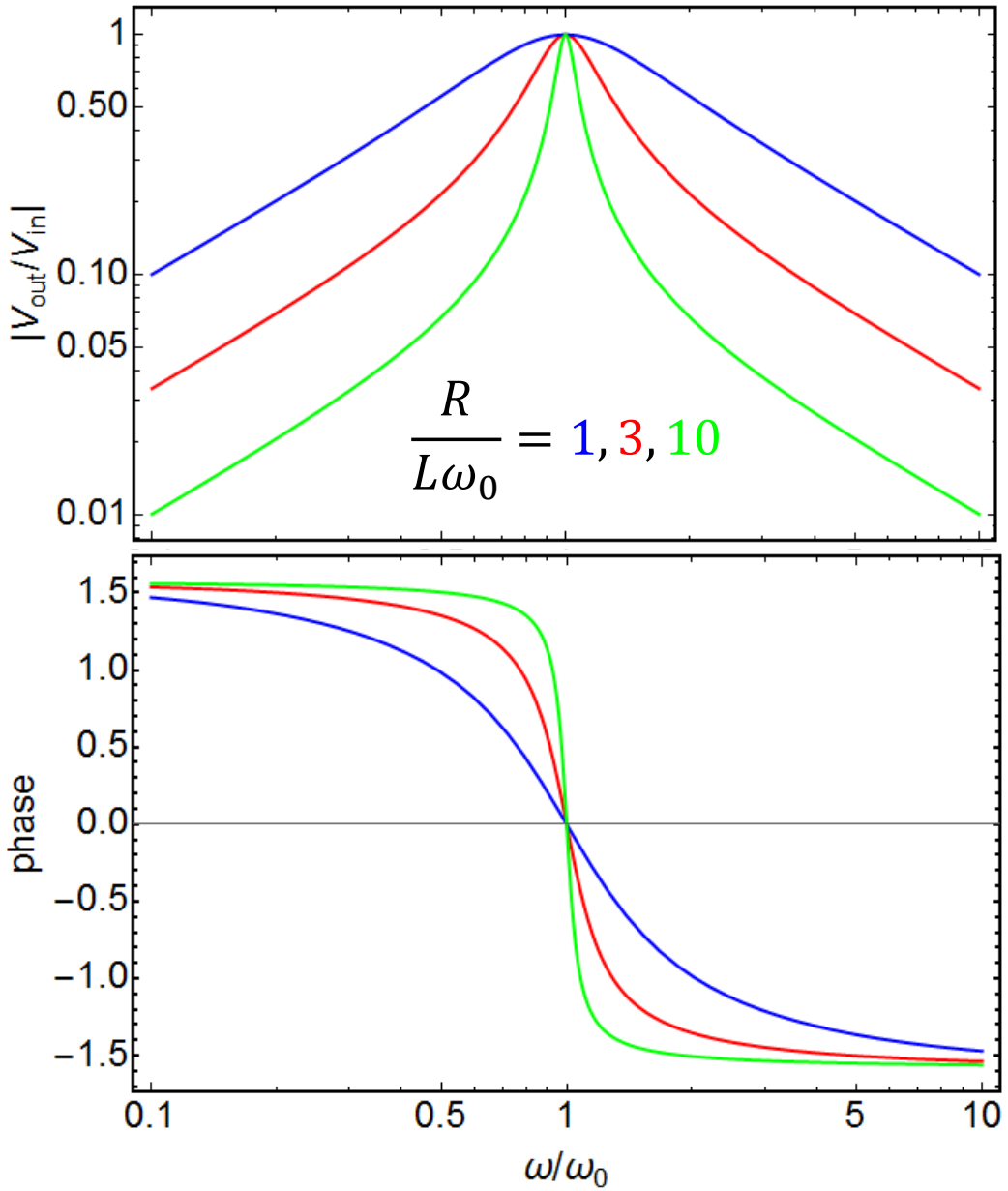
Quality factor

$$Q = \frac{\omega_0}{\Delta\omega_{FWHM}} = \frac{R}{L\omega_0} = R\sqrt{\frac{C}{L}}$$

Bandpass filter – Bode plots

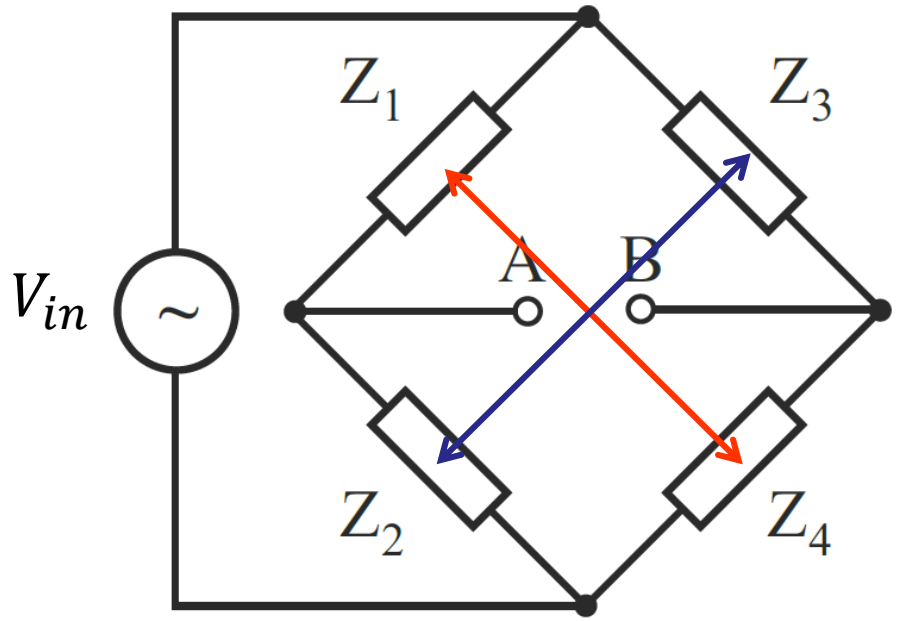
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+\left(\frac{R}{L\omega_0}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right)^2}}$$

$$\phi = -\arctan\left(\frac{R}{L\omega_0}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right)$$



Bridge circuits

Used to determine an unknown impedance



$$V_A = \frac{Z_2}{Z_1 + Z_2} V_{in}$$

$$V_B = \frac{Z_4}{Z_3 + Z_4} V_{in}$$

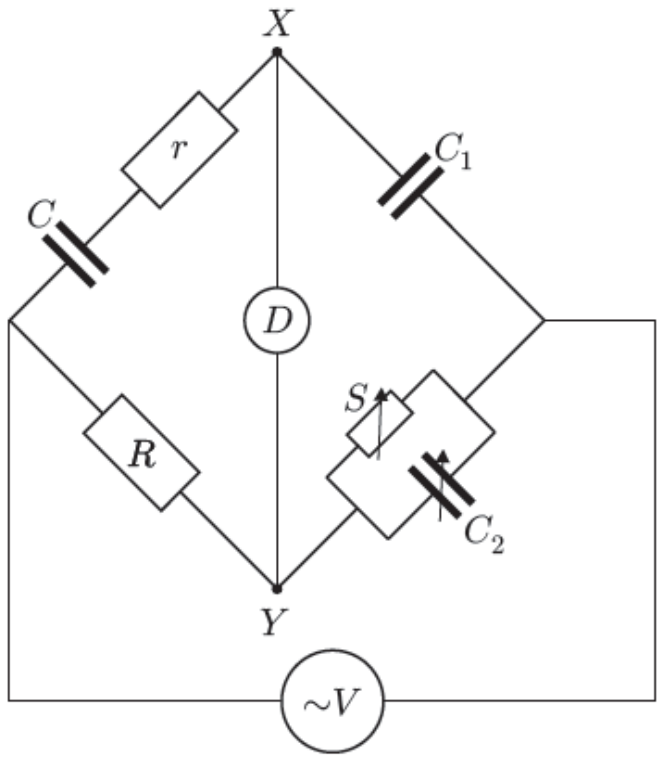
Bridge balanced when $V_{AB} = 0 \rightarrow \frac{Z_2}{Z_1 + Z_2} = \frac{Z_4}{Z_3 + Z_4}$

$$\rightarrow Z_1 Z_4 = Z_2 Z_3$$

(gives 2 equations – real and imaginary parts)

CP2 September 2003 – part (i)

9. (i) A Schering bridge is used to determine the capacitance C and resistance r of a lossy capacitor. In the diagram below the instrument D detects when X and Y are at the same potential, i.e. the bridge is balanced.



Show that the conditions for balancing the bridge are

$$C = C_1 S/R \quad \text{and} \quad r = C_2 R/C_1 \,. \qquad [10]$$

Power in AC circuits

Power into component: $P = VI$ (real values)

So remember $P = \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V})$

$$P = I_0 \cos(\omega t) \times I_0 |Z| \cos(\omega t + \phi)$$

Time average $\rightarrow \langle P \rangle = \frac{1}{2} I_0^2 |Z| \cos(\phi) = \frac{1}{2} I_0^2 \mathrm{Re}(Z)$

Power factor

Or $\rightarrow \langle P \rangle = \frac{1}{2} V_0 I_0 \cos(\phi) = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos(\phi)$

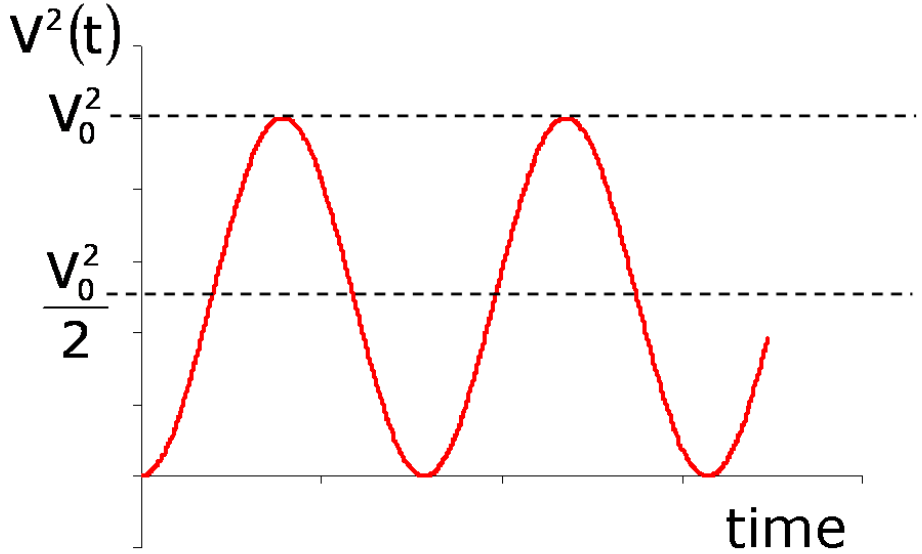
Resistive load $Z = R, \cos(\phi) = 1$

Reactive load $Z = iX, \cos(\phi) = 0$

RMS values

For resistive load $\langle P \rangle = \langle IV \rangle = \left\langle \frac{V^2}{R} \right\rangle = \frac{V_{\mathrm{rms}}^2}{R}$

$$V_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T V^2(t)\,dt}$$



For sinusoid $V_{\mathrm{rms}} = \frac{V_0}{\sqrt{2}}$

CP2 September 2003 – part (ii)

(ii) A source of alternating voltage $V_0 \exp(j\omega t)$ with internal complex impedance $Z_1 = R_1 + jX_1$ is connected to a load of complex impedance $Z_2 = R_2 + jX_2$.

Show that the current through the load is given by

$$I = I_0 \exp[j(\omega t - \phi)],$$

where

$$I_0 = V_0 \left[(R_1 + R_2)^2 + (X_1 + X_2)^2\right]^{-1/2},$$

and find an expression for ϕ. [4]

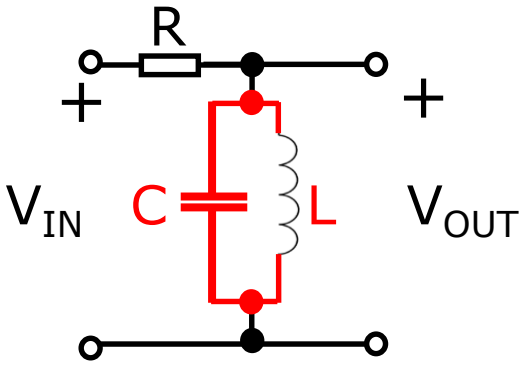
Hence show that the mean power $\langle P \rangle$ consumed by the load over one cycle is

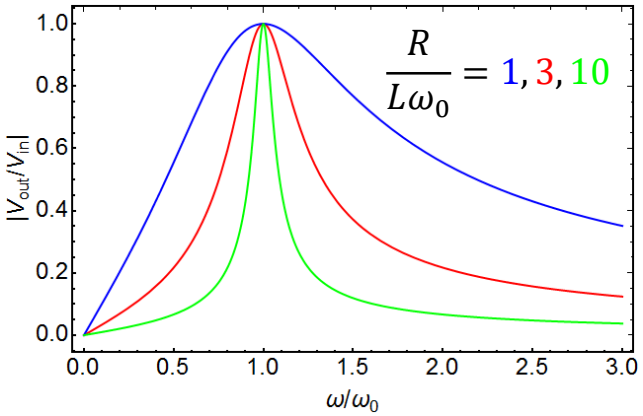
$$\langle P \rangle = \tfrac{1}{2} I_0^2 R_2. \quad [6]$$

Circuit Theory
Lecture 8

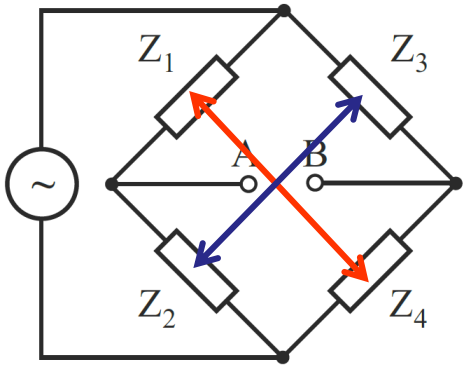
Recap

- Band pass filter





- Bridge circuits



Balanced when

$$Z_1 Z_4 = Z_2 Z_3$$

- Power dissipation in AC circuits

$$\langle P \rangle = \frac{1}{2} I_0^2 \mathrm{Re}(Z) = \frac{1}{2} V_0 I_0 \cos(\phi) = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos(\phi)$$

$$Z = |Z| e^{i\phi}$$

CP2 September 2003 – part (ii)

(ii) A source of alternating voltage $V_0 \exp(j\omega t)$ with internal complex impedance $Z_1 = R_1 + jX_1$ is connected to a load of complex impedance $Z_2 = R_2 + jX_2$.

Show that the current through the load is given by

$$I = I_0 \exp[j(\omega t - \phi)],$$

where

$$I_0 = V_0 \left[(R_1 + R_2)^2 + (X_1 + X_2)^2\right]^{-1/2},$$

and find an expression for ϕ. [4]

Hence show that the mean power $\langle P \rangle$ consumed by the load over one cycle is

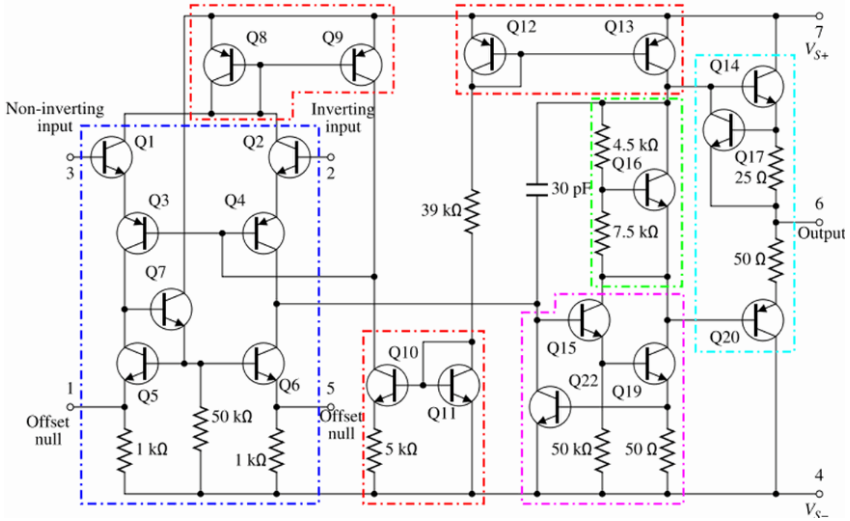
$$\langle P \rangle = \tfrac{1}{2} I_0^2 R_2. \quad [6]$$

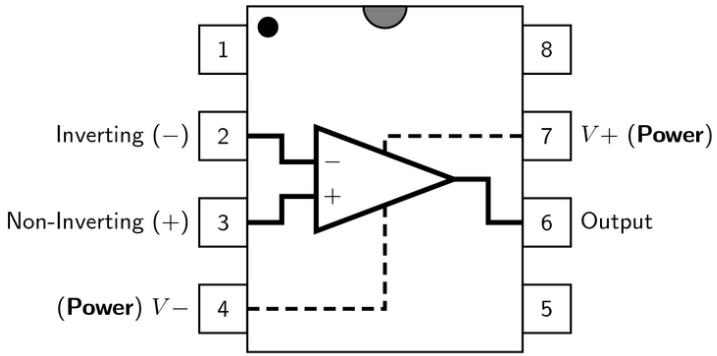
Today

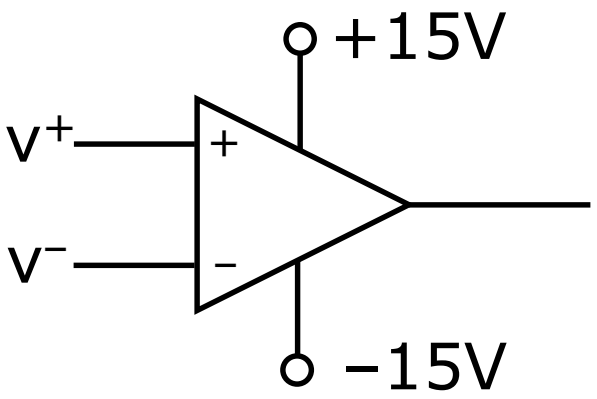
- Op-amps

Operational amplifiers (op-amps)

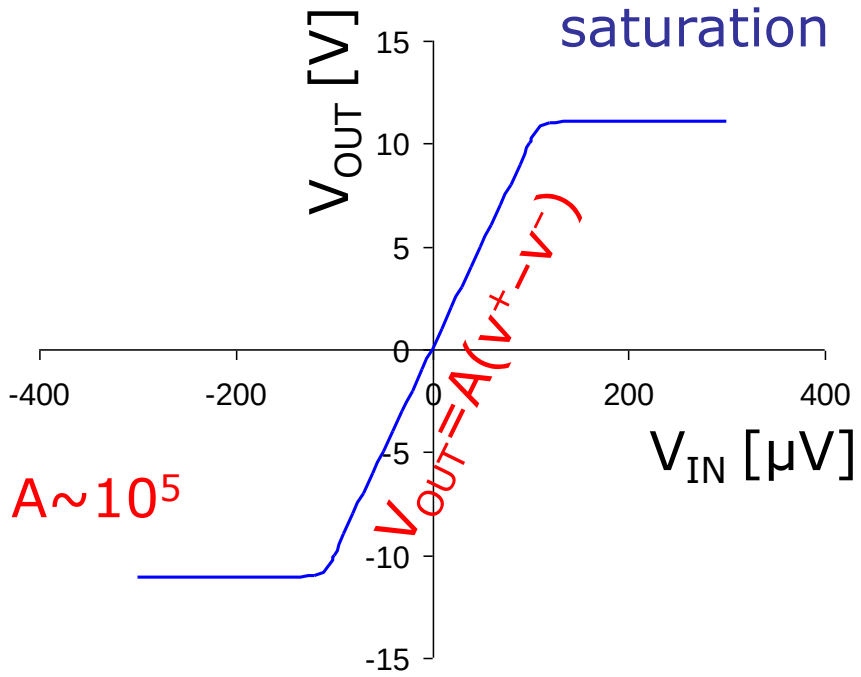
- An op-amp is an active (powered) device that amplifies a voltage difference by a large factor



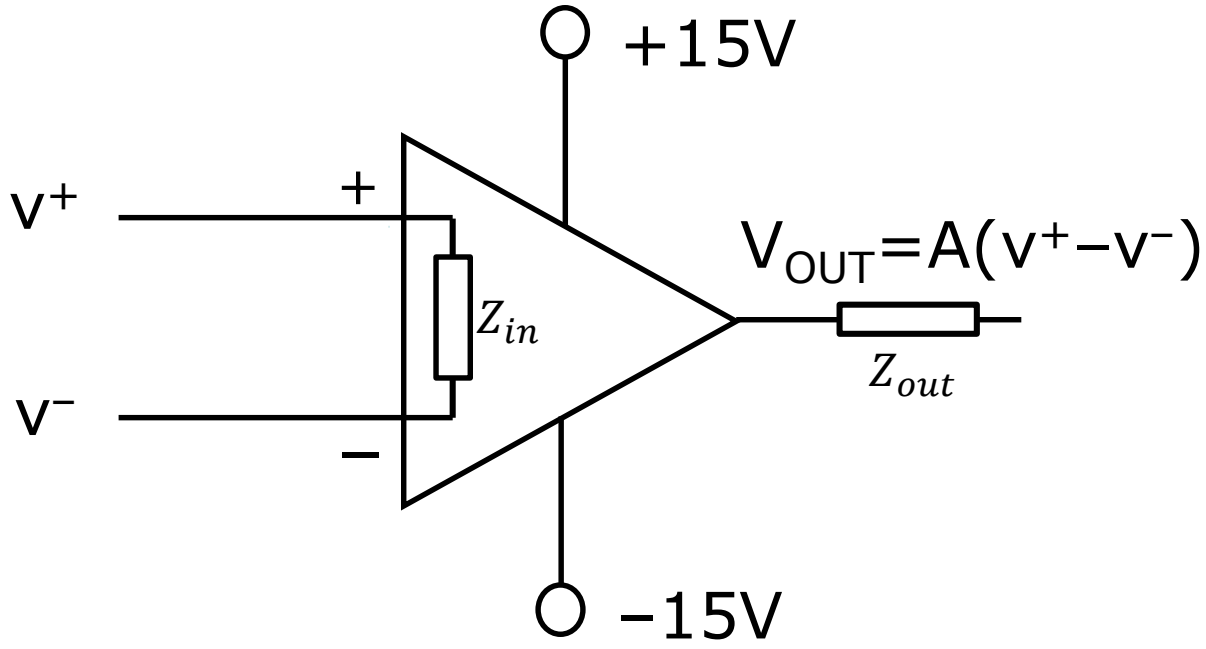




$$V_{out} = A(V^+ - V^-)$$



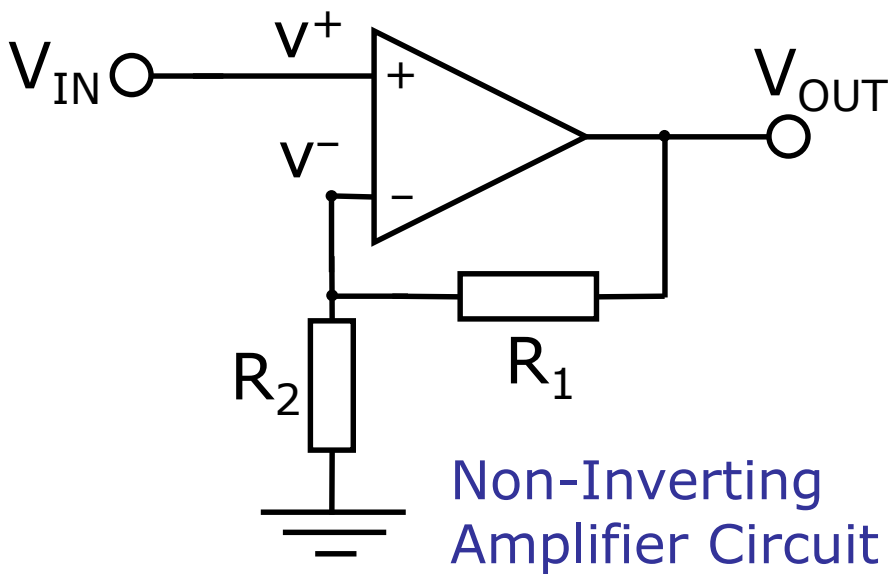
The (ideal) op-amp



Real		Ideal	
$Z_{in} \sim 10^7\ \Omega$		$Z_{in} = \infty$	(draws no current)
$Z_{out} \sim 100\ \Omega$	→	$Z_{out} = 0$	(supplies unlimited current)
$A \sim 10^5$		$A = \infty$	$(v_+ = v_-)$

Need feedback

Op-amp Feedback



Non-Inverting Amplifier Circuit

$$V_{OUT} = A(v^+ - v^-)$$

$$V_{OUT} = A\left(V_{IN} - V_{OUT}\frac{R_2}{R_1 + R_2}\right)$$

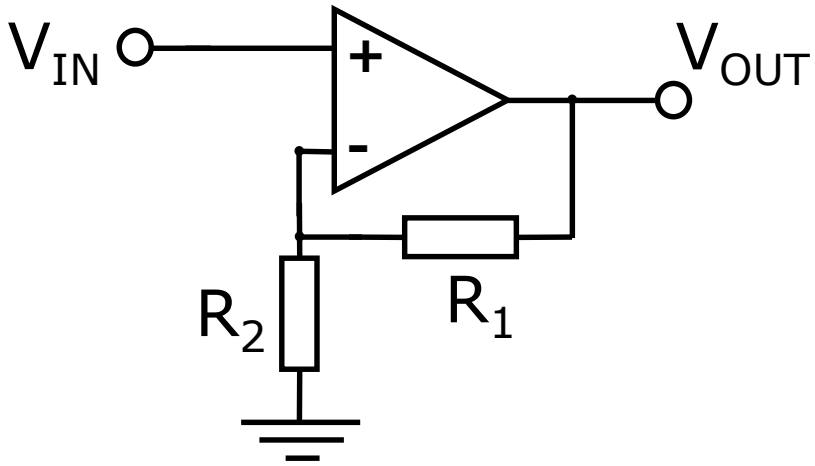
$$V_{OUT} = A\left(V_{IN} - V_{OUT}\frac{R_2}{R_1 + R_2}\right)$$

$$V_{OUT}\left(\frac{1}{A} + \frac{R_1}{R_1 + R_2}\right) = AV_{IN}$$

$$\frac{V_{OUT}}{V_{IN}} = \frac{1}{\left(\frac{1}{A} + \frac{R_1}{R_1 + R_2}\right)} \cong \frac{R_1 + R_2}{R_2}$$

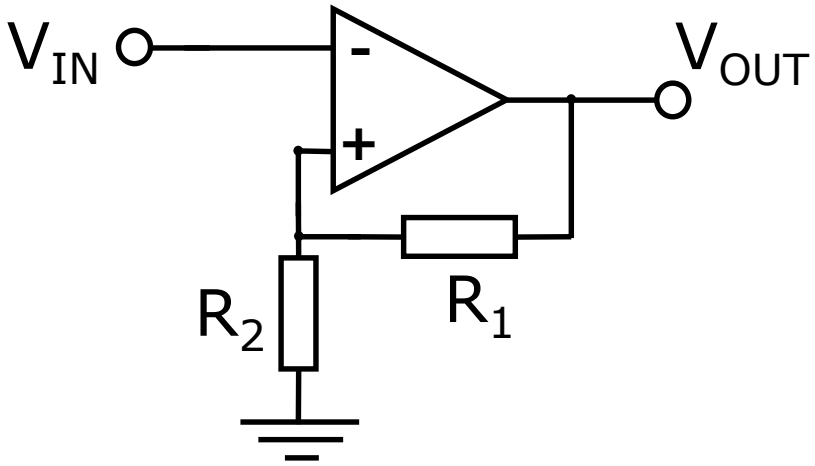
But feedback must be negative

Negative feedback



$$V_{\text{out}} \uparrow \;\mapsto\; V_{-} \uparrow \;\mapsto\; V_{\text{out}} \downarrow$$

Negative feedback
stable



$$V_{\text{out}} \uparrow \;\mapsto\; V_{+} \uparrow \;\mapsto\; V_{\text{out}} \uparrow$$

Positive feedback
unstable

Op-amp golden rules

- Golden Rule #1: the inputs draw no current

  Because $Z_{in} = \infty$

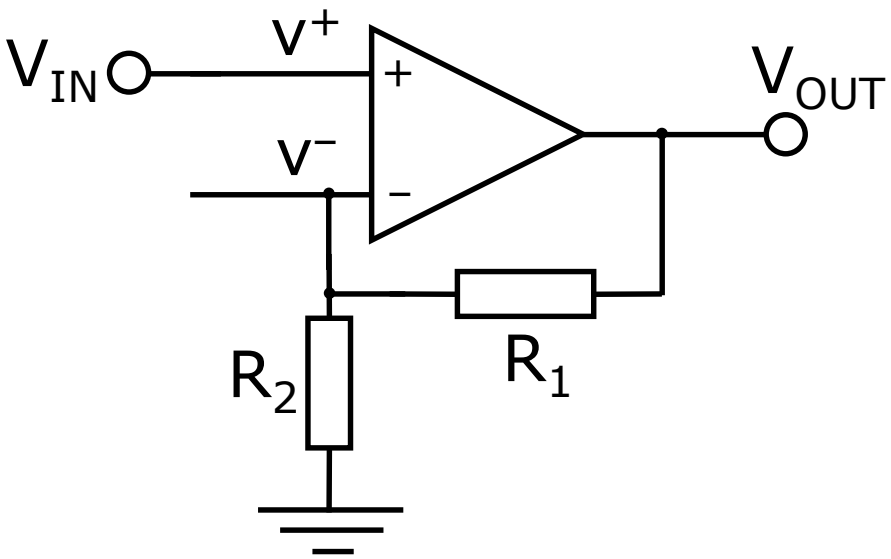
- Golden Rule #2: $V_+ = V_-$

  Because $A = \infty$

  This requires negative feedback

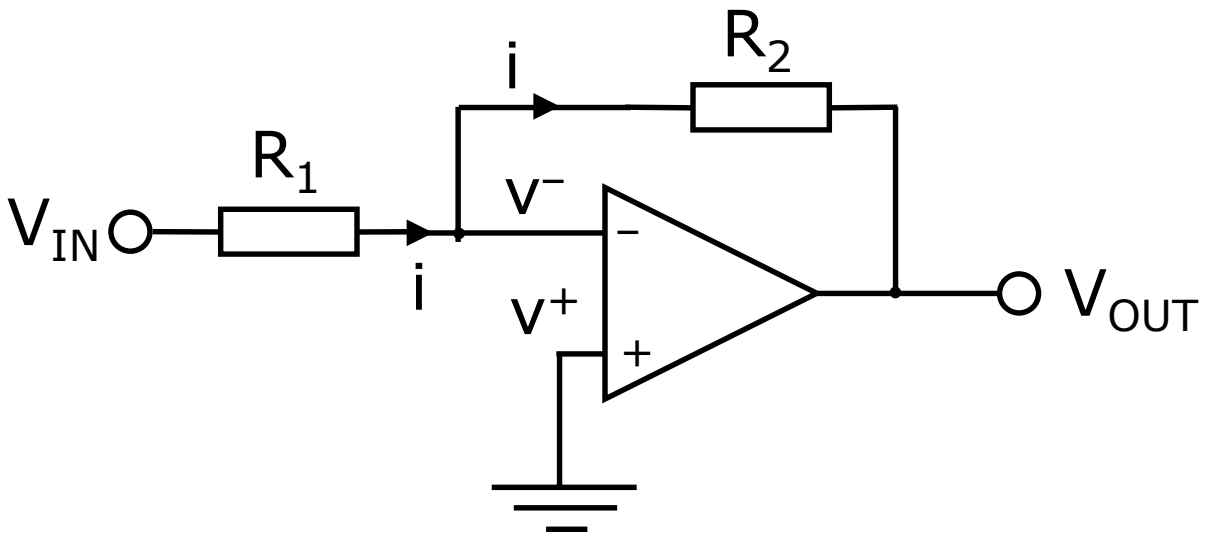
Applying these means you can analyse a circuit containing an op-amp without knowing anything about the details of the op-amp

Non-Inverting Amplifier (again)



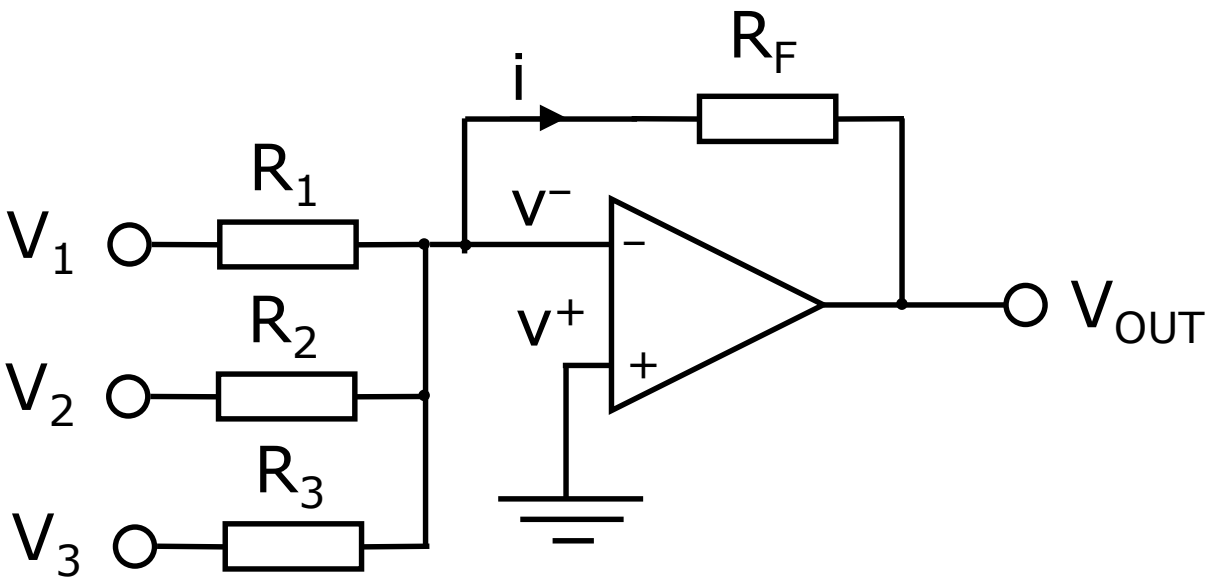
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R_1 + R_2}{R_1}$$

Inverting Amplifier



$$\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}$$

Summing Amplifier



$$V_{\text{out}} = -R_F\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right)$$

Next lecture

- More examples
- Exploiting positive feedback and saturation

Circuit Theory
Lecture 9

Recap: Op-amp golden rules

- Golden Rule #1: the inputs draw no current
  Because $Z_{in} = \infty$

- Golden Rule #2: $V_+ = V_-$
  Because $A = \infty$
  This requires negative feedback

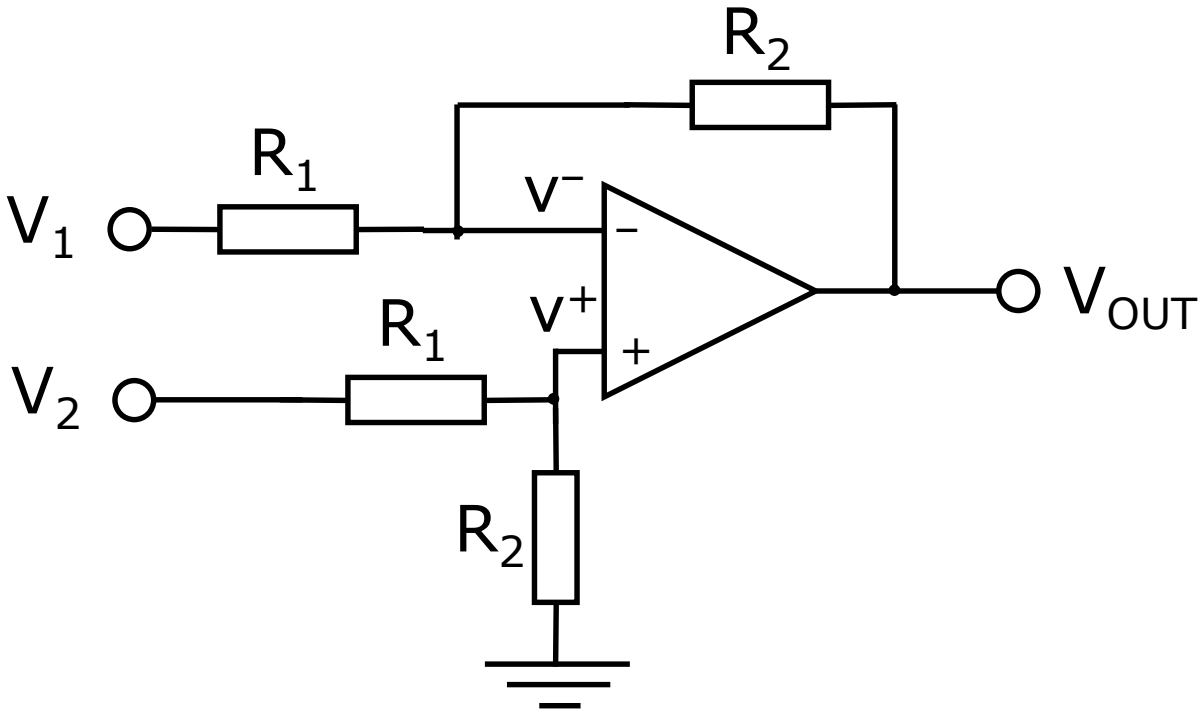
Applying these means you can analyse a circuit containing an op-amp without knowing anything about the details of the op-amp

Today

More examples

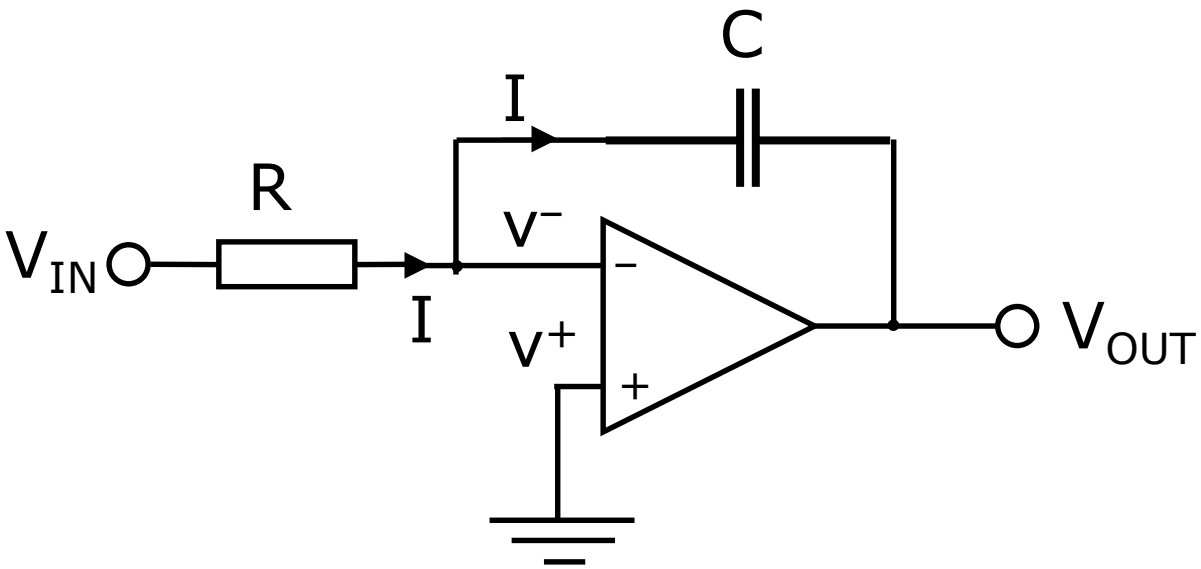
- Difference amplifier
- Integrator
- Differentiator
- AC analysis
- Using saturation

Difference amplifier



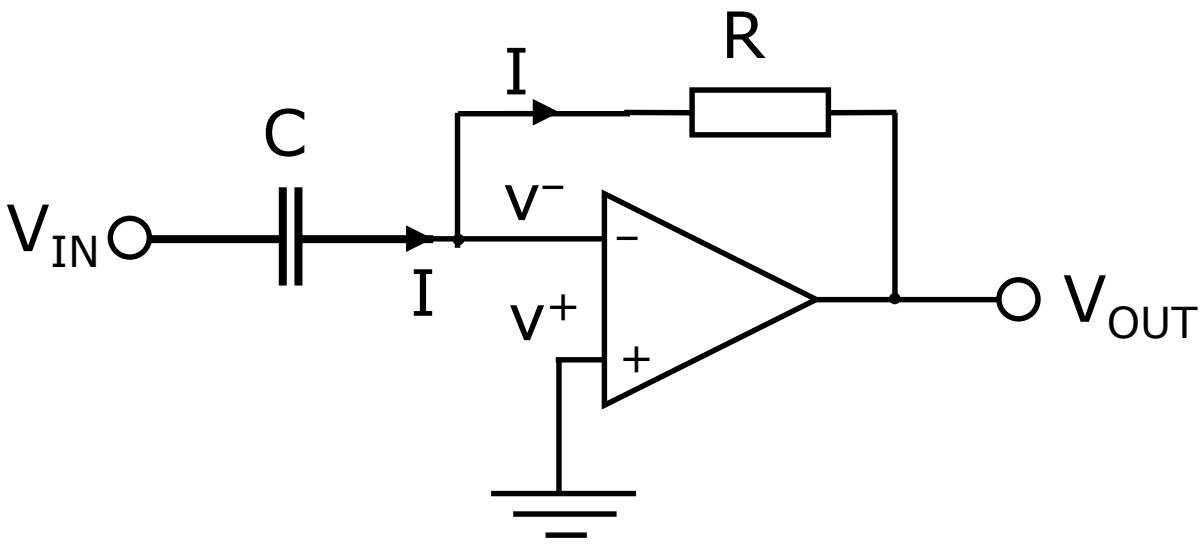
$$V_{out} = \frac{R_2}{R_1}(V_2 - V_1)$$

Op-amp Integrator



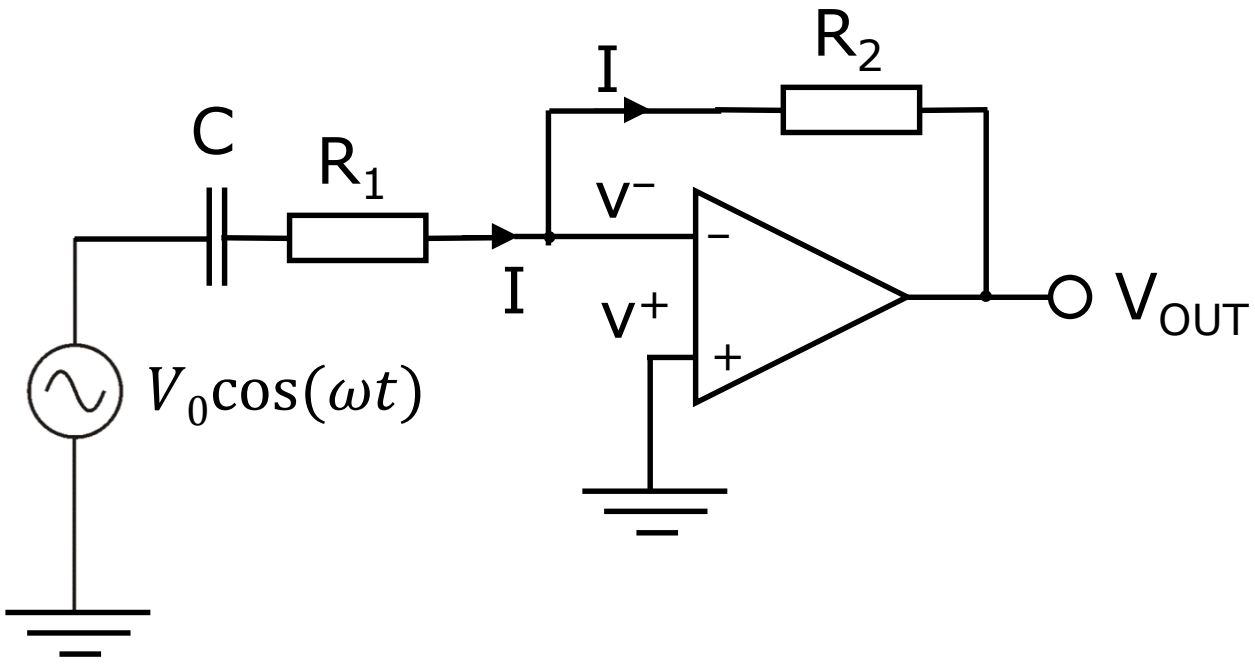
$$V_{out} = -\frac{1}{RC}\int_0^t V_{in}\,dt$$

Op-amp Differentiator

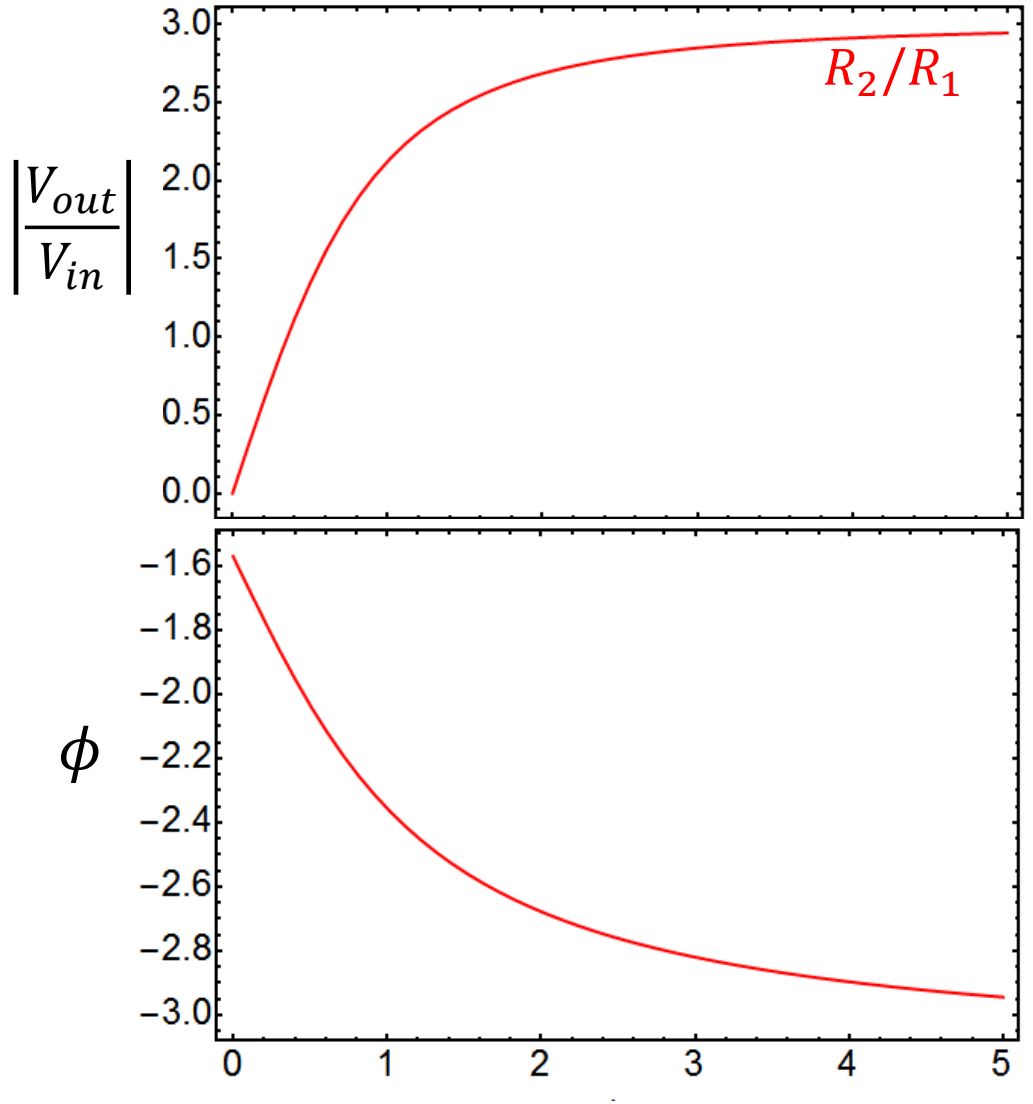


$$V_{out} = -RC\frac{dV_{in}}{dt}$$

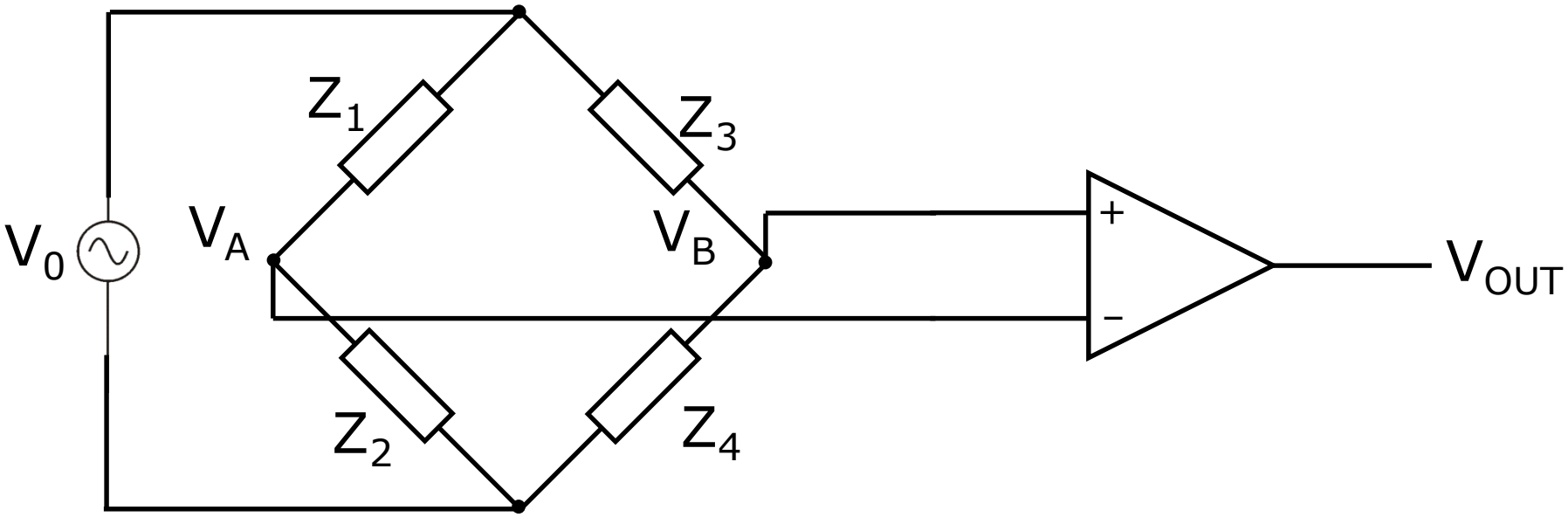
Complex analysis



$$\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1 + \frac{1}{i\,\omega C}}$$



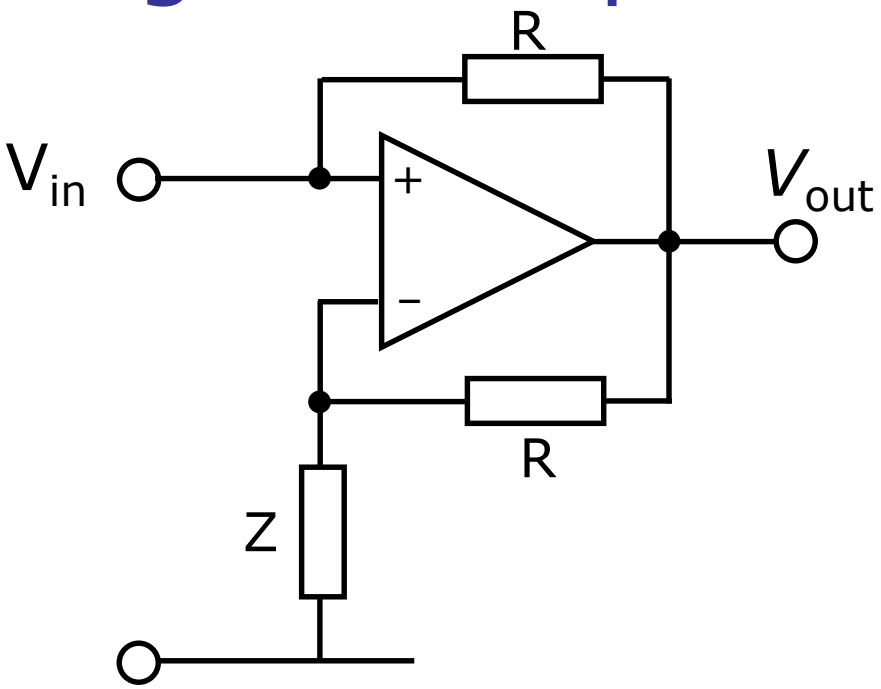
Bridge circuits



Bridge balanced when $V_A - V_B = 0$

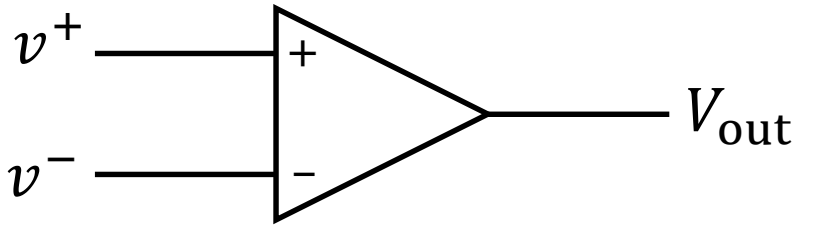
$$Z_2 Z_3 = Z_4 Z_1$$

Negative Impedance Converter



$$Z_{in} = \frac{V_{in}}{I_{in}} = -Z$$

Exploiting op-amp saturation



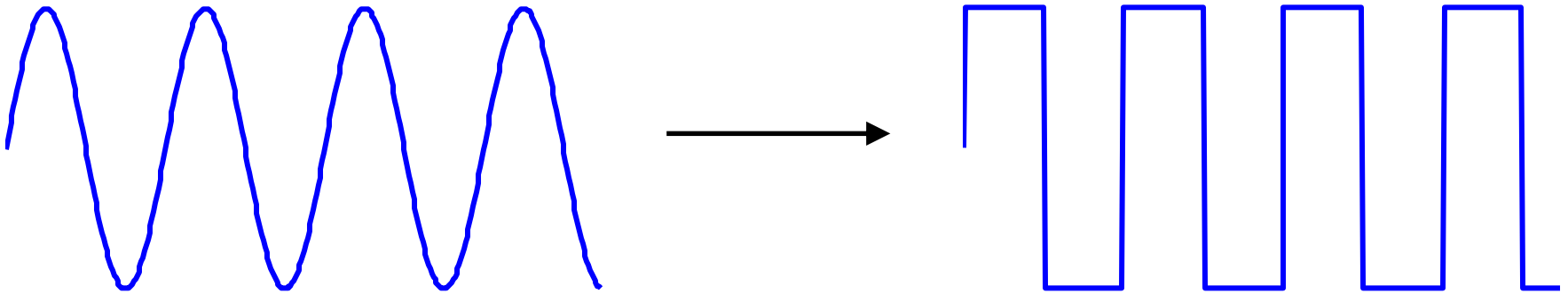
$A \to \infty$

$$V_{\text{out}} = A(v^+ - v^-)$$

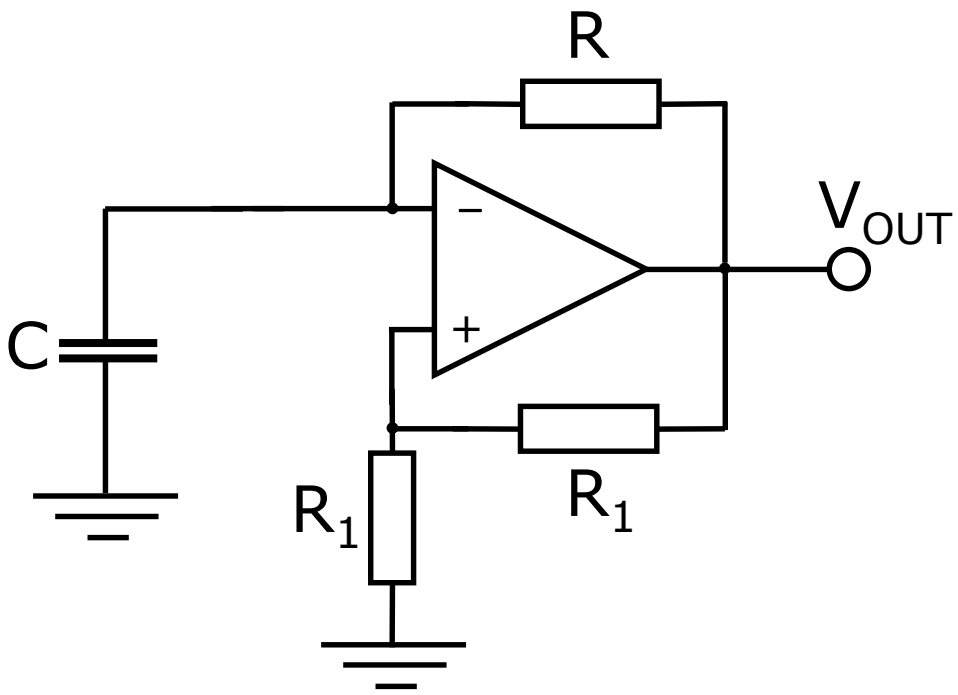
$$V_{\text{out}} = +V_{\text{sat}} \text{ for } v^+ > v^-$$

$$V_{\text{out}} = -V_{\text{sat}} \text{ for } v^+ < v^-$$

Saturation voltage



Oscillator (non ideal)



What does this circuit do?

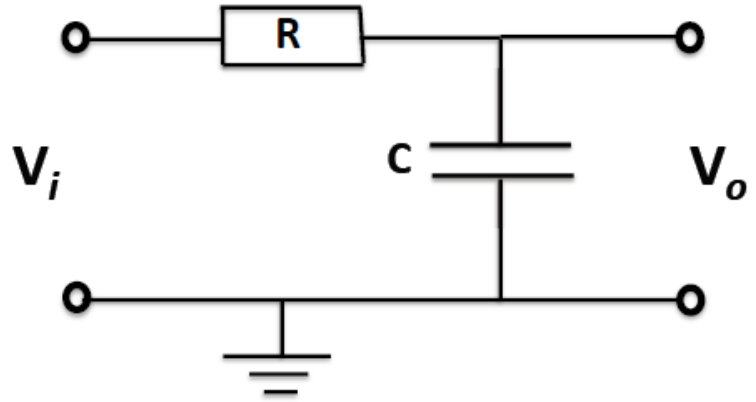
Op-amp applications

Building Block of analogue electronics

- Signal amplifiers
- Audio amplifiers
- Integrators / differentiators
- Voltage / current sources
- Active filters
- Oscillators
- Digital-analogue and analogue-digital convertors

CP2 2019

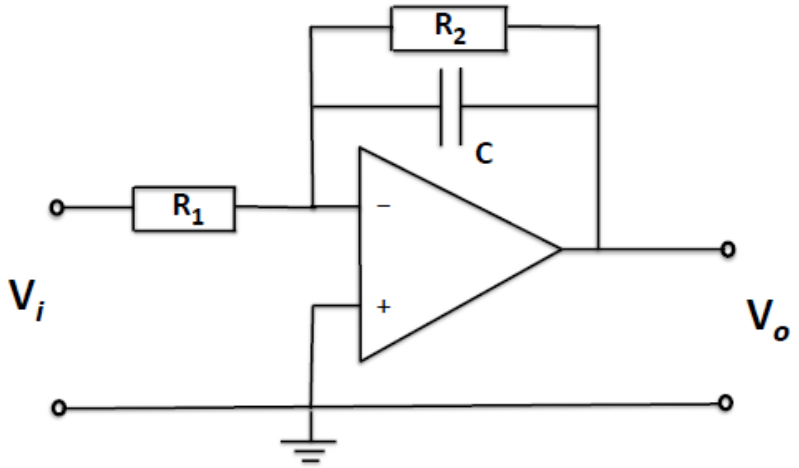
10. A circuit consists of a resistor R and a capacitor C together with an input voltage V_i and is illustrated in the figure below. The input voltage V_i is a sinusoidal function of time. Find an expression for the magnitude and phase of the output voltage V_o. The resistor has a resistance of 5 kΩ and the capacitor has a capacitance of 3.2 nF. Determine the behaviour of the circuit as a function of frequency and find the output voltage at 100 Hz, 10^4 Hz and 5×10^4 Hz. Sketch a plot of the amplitude of V_o as a function of frequency. Give an application for such a circuit. [10]



CP2 2019

An alternative design employs an operational amplifier and is illustrated below. The signal is transmitted to the inverting input of the op-amp through resistor R_1, while a capacitor C in parallel with a resistor R_2 connects the input and output of the op-amp. Again the input voltage V_i is a sinusoidal function of time. Find an expression for the output voltage V_o.

The resistor R_1 has a resistance of 1 kΩ, the resistor R_2 has a resistance of 5 kΩ, and the capacitor C has a capacitance of 3.2 nF. How does the output from this circuit compare with that from the circuit above? [7]



Comment on the advantages and disadvantages of the two circuits. [3]

Summary

- Op-amps are building blocks of analogue electronics
- When used with negative feedback can usually treat as ideal and so
  - Golden Rule #1: the inputs draw no current
    Because $Z_{in} = \infty$
  - Golden Rule #2: $V_+ = V_-$
    Because $A = \infty$
    This requires negative feedback

Thanks for your attention