Multivariate Calculus

(Multiple Integrals and Vector Calculus)

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Problem Sheet 4: "Manifolds, integration, Gauß's and Stokes's theorems"

Question 1. (Surfaces and minimal distances)

A hyper-surface $M \subset \mathbb{R}^n$ is (locally) described by a parametrisation $X : U \to M \cap V$ or the zero locus of a function $f : \mathbb{R}^n \to \mathbb{R}$.

a) Find conditions for the distance of points $x \in M$ from the origin to be extremal. Formulate these conditions both in terms of a parametrisation X and in terms of the function f.

b) Apply your results from part a) to the surface $M \subset \mathbb{R}^3$ defined by the zero locus of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = \frac{1}{2}(x^2 + y^2) - z - 1$, in order to find the points on M closest to the origin.

c) Do the same for the surface $M \subset \mathbb{R}^3$ defined by the zero locus of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = \frac{1}{3}(x^3 + y^3 + z^3) - 1$.

Question 2. (Surface integrals and Gauß's theorem)

Evaluate the surface integral $\int_M A \cdot dS$ for the following surfaces $M \subset \mathbb{R}^3$ and vector fields $A : \mathbb{R}^3 \to \mathbb{R}^3$. If appropriate, check your result using Gauß's theorem.

a) $A(x, y, z) = (y, 2x, -z)^T$ and the surface M given by the parametrisation $X(t_1, t_2) = (t_1, 6-2t_1, t_2)^T$, where $t_1 \in [0, 3]$ and $t_2 \in [0, 4]$.

b) $A(x, y, z) = (x + y^2, -2x, 2yz)^T$ with surface $M = \{(x, y, z)^T \in \mathbb{R}^3 \,|\, 2x + y + 2z = 6 \text{ and } x, y, z \geq 0\}$.

c) $A(x, y, z) = (6z, 2x + y, -x)^T$ and $M = \partial Q$ where $Q = \{(x, y, z)^T \in \mathbb{R}^3 \,|\, x^2 + z^2 \leq 9 \text{ and } x, z \geq 0 \text{ and } y \in [0, 8]\}$.

d) $A(x, y, z) = (xy^2, x^2, yz)^T$ and $M = \partial P$ where $P = \{(x, y, z)^T \in \mathbb{R}^3 \,|\, x, y \geq 0 \text{ and } x + y \leq 1 \text{ and } z \in [0, 1]\}$.

Question 3. (Gauß's theorem on a cone)

A cone C with height h and the base a disk with radius R is given by

$$C = \left\{ \left((1 - \frac{z}{h}) \, r \cos(\varphi), \left((1 - \frac{z}{h}) \, r \sin(\varphi), z \right)^T \in \mathbb{R}^3 \,|\, r \in [0, R],\ \varphi \in [0, 2\pi),\ z \in [0, h] \right\} \right. .$$

The vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $A(x, y, z) = (x, y, x^2)^T$.

a) What is the boundary of C? Work out the measures $d\sigma$ and dS on ∂C, using the outer unit normal vector.

b) Compute $\text{vol}(C)$ and $\text{vol}(\partial C)$.

c) Work out the Laplacian Δ on ∂C and find the solutions to $\Delta f = 0$ which are independent of the azimutal angle.

d) Check Gauß's theorem for the vector field A and the cone C.

Question 4. (Stokes's theorem)

Verify Stoke's theorem for the following examples.

a) The vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A(x, y, z) = (y, -x, z)^T$ and the upper half-sphere $S_+ = \{(x, y, z)^T \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$.

b) The vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A(x, y, z) = (-y, xz, z)^T$ and the cylinder $C = \{(\cos(\varphi), \sin(\varphi), z)^T \in \mathbb{R}^3 \,|\, \varphi \in [0, 2\pi) \text{ and } z \in [0, 1]\}$.

Question 5. (Computing some areas and volumes)

a) Let $M \subset \mathbb{R}^n$ be a hyper-surface with finite volume and $\lambda M = \{\lambda x \,|\, x \in M\}$ the re-scaled version of the hyper-surface with scale factor $\lambda \in \mathbb{R}$. Find a relation between $\text{vol}(M)$ and $\text{vol}(\lambda M)$.

b) Derive a relation between the volume of B_1^n, the n-dimensional ball with radius 1, and the volume of S_1^{n-1}, the $(n-1)$-dimensional sphere with radius one. How does the volume of B_R^n (the n-dimensional ball with radius R) relate to the volume of S_R^{n-1} (the $(n-1)$-dimensional sphere with radius R)?

c) Let $\mathcal{S} \subset \mathbb{R}^2$ be a set with finite area and boundary $C = \partial\mathcal{S}$. Derive a formula for the area of $\mathcal{S}$ in terms of a curve integral over C.

d) A set $\mathcal{S} \subset \mathbb{R}^2$ is bounded by a polygon with vertives $v_1, \ldots, v_n, v_{n+1} = v_1 \in \mathbb{R}^2$. Derive a formula for the area of $\mathcal{S}$ in terms of the vertex vectors v_a.