Multivariate Calculus

(Multiple Integrals and Vector Calculus)

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Problem Sheet 3: "Integration in $\mathbb{R}^n$"

Question 1. (Two-dimensional integrals)

For examples a), b) and c), sketch the integration regions and carry out the integrals in two ways, by choosing different splittings into one-dimensional integrals. For each case, make sure that both calculation give the same answer. For part d) split into one-dimensional integrals so that the integral can be carried out analytically.

a) $I_i = \int_{S_1} dx\, dy\, y$ where $S_1 = \{(x,y)^T \in \mathbb{R}^2 \,|\, 0 \leq y \leq \sqrt{2} \text{ and } y^2 \leq x \leq 2\}$.

b) $I_2 = \int_{S_2} dx\, dy\, \sqrt{x}y$ where $S_2 = \{(x,y)^T \in \mathbb{R}^2 \,|\, 0 \leq x \leq 4 \text{ and } 0 \leq y \leq \sqrt{x}\}$.

c) $I_3 = \int_{S_3} dx\, dy\, x$ where $S_3 = \{(x,y)^T \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1 \text{ and } -y \leq x \leq y^2\}$.

d) $I_4 = \int_{S_4} dx\, dy\, x^{-1} \sin(x)$ where $S_4 = \{(x,y)^T \in \mathbb{R}^2 \,|\, 0 \leq y \leq \pi \text{ and } y \leq x \leq \pi\}$.

Question 2. (Transformation formula)

Carry out the following integrals by transforming to a suitable new set of coordinates first.

a) $I_1 = \int_{S_1} dx\, dy\, (x^2+y^2) \arctan(y/x)$ where $S_1 = \{(x,y)^T \in \mathbb{R}^2 \,|\, 0 \leq y \leq a \text{ and } 0 \leq x \leq \sqrt{a^2-y^2}\}$, where $a \in \mathbb{R}$ is a positive constant.

b) $I_2 = \int_{S_2} dx\, dy\, (x^2+y^2)$ where $S_2 = \{ua+vb \in \mathbb{R}^2 \,|\, 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 1\}$ with vectors $a = (1,1)^T$ and $b = (-1,1)^T$.

c) $I_3 = \int_{S_3} dx\, dy\, \exp(-xy)$ where $S_3 = \{(x,y) \in \mathbb{R}^2 \,|\, x,y \geq 0 \text{ and } xy \leq 1 \text{ and } 1/2 \leq y/x \leq 2\}$.

d) $I_4 = \int_{S_4} dx\, dy\, dz (zx^2+y^2) e^{-z}$ where $S_4 = \{(x,y,z)^T \in \mathbb{R}^3 \,|\, z \in [0,\infty) \text{ and } x^2+y^2 \leq R^2\}$, where $R \in \mathbb{R}$ is a constant.

Question 3. (Ellipsoids)

The interior $E_n(a_1,\ldots,a_n) \subset \mathbb{R}^n$ of an ellipsoid in n dimensions with (positive) half axes of lenght $a_1,\ldots,a_n$ is defined by

$$E_n(a_1,\ldots,a_n) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \frac{x_i^2}{a_i^2} \leq 1 \right\} .$$

a) Begin with the two-dimensional case and work out the area of an ellipse interior $E_2(a,b)$ with half axes lengths a, b.

b) For the three-dimensional case, work out the volume of an ellipsoid interior $E_3(a,b,c)$ with half axes lengths a, b, c.

c) Let $\mathcal{S} \subset \mathbb{R}^n$ be a set and P an $n \times n$ matrix. The linearly transformed set $P\mathcal{S}$ is defined as $P\mathcal{S} = \{Px \,|\, x \in \mathcal{S}\}$. Work out how the volumes of $\mathcal{S}$ and $P\mathcal{S}$ relate. Specialise your result to the case of a re-scaling with a factor $\lambda \in \mathbb{R}$, that is, to a matrix $P = \lambda \mathbb{1}_n$.

d) Find a recursive relation for $E_n(a_1, \ldots, a_n)$ in terms of $E_{n-1}(a_1, \ldots, a_{n-1})$. Iterate this relation to find an explicit formula for the volume of $E_n(a_1, \ldots, a_n)$ for any dimension n.

e) For a positive definite $n \times n$ matrix Q define the set $E_n(Q) = \{x \in \mathbb{R}^n \,|\, x^T Q x = 1\}$. Compute the volume of $E_n(Q)$.

Question 4. (Symmetries, centre of mass and moment of inertia tensor)
In $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ consider a mass density function $\rho : \mathbb{R}^3 \to \mathbb{R}^{\geq 0}$ with compact support. This function defines a massive object $\mathcal{S} = \text{supp}(\rho)$. The total mass M, the centre of mass $\bar{x}_i$ and the moment of inertia tensor I_{ij} for $\mathcal{S}$ are defined as

$$M = \int_{\mathbb{R}^3} d^3x\, \rho(x)\,, \qquad \bar{x}_i = \frac{1}{M}\int_{\mathbb{R}^3} d^3\, \rho(x) x_i\,, \qquad I_{ij} = \int_{\mathbb{R}^3} d^3x\, \rho(x)\left(|x|^2\delta_{ij} - x_i x_j\right)\,.$$

a) Given a mass density function ρ define a new transformed mass density function $\tilde{\rho}$ by $\tilde{\rho}(x) = \rho(Rx)$, where R is a 3×3 orthogonal matrix. Work out the relationships between the masses M and $\tilde{M}$, the centre of masses $\bar{x}_i$ and $\tilde{\bar{x}}_i$ and the moment of inertia tensors I_{ij} and $\tilde{I}_{ij}$ of the old and the transformed object.

b) Assume that the object described by ρ is invariant under the transformation generated by R from part a), that is, $\tilde{\rho} = \rho$. What does this imply for the centre of mass and the moment of inertia tensor?

c) Suppose a mass density function ρ is invariant under all three-dimensional rotations. What are the most general forms of the centre of mass and the moment of inertia tensor consistent with this symmetry? What is the most general form of the centre of mass and the moment of inertia tensor if ρ is invariant only under rotations around the z-axis?

d) Consider a mass density

$$\rho(x) = \begin{cases} \rho_0 & \text{for} \quad |x| \leq R \\ 0 & \text{for} \quad |x| > R \end{cases}\,, \tag{1}$$

where ρ_0 is a constant, whose support is a ball $B_R(0)$ around the origin with radius R. Use the results from parts a), b), c) to constrain the centre of mass and moment inertia tensor as much as possible. Fix the remaining entries by an explicit calculation. Do the same for a mass density

$$\rho(x) = \begin{cases} \rho_0 & \text{for } |x| \leq R \text{ and } x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases}\,, \tag{2}$$

where $\rho_0 \in \mathbb{R}$ is a constant, whose support is an upper half of a ball $B_R^+(0)$ with radius R.

e) Consider a cone and associated mass density

$$C = \left\{ \left(\left(1 - \frac{z}{h}\right) r\cos(\varphi), \left(1 - \frac{z}{h}\right) r\sin(\varphi), z \right)^T \in \mathbb{R}^3 \;\middle|\; r \in [0, R],\ z \in [0, h],\ \varphi \in [0, 2\pi) \right\}$$

$$\rho(x) = \begin{cases} \rho_0 \frac{z}{h} & \text{for } x \in C \\ 0 & \text{for } x \notin C \end{cases}$$

Find the mass, the centre of mass and the moment of inertia tensor for this cone.

Question 5. (Gaußian integrals and beyond)

A probability density $p : \mathbb{R}^n \to \mathbb{R}$ is given by $p(x) = N \exp(-\frac{1}{2} x^T A x)$, where $N \in \mathbb{R}$ is a normalisation factor and A is a symmetric, positive definite $n \times n$ matrix. The expectation value of a function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to p is defined as

$$\langle f \rangle = \int_{\mathbb{R}^n} d^n x\, f(x) p(x) \,.$$

a) Determine the normalisation constant so that $\langle 1 \rangle = 1$.

b) Compute the expectation value $Z(b) := \langle \exp(b \cdot x) \rangle$ and express your result in terms of the matrix $\Delta = A^{-1}$.

c) The k^{th} moments of p are defined as the expectation values $\langle x_{i_1} \cdots x_{i_k} \rangle$. Show that

$$\langle x_{i_1} \cdots x_{i_k} \rangle = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_k}} Z \bigg|_{b=0} \,.$$

d) Show that all odd moments of p vanish and compute the second and fourth moments, using the previous results. Also compute $\langle x_i x_j \exp(\lambda \sum_k x_k^4) \rangle$, where $\lambda \in \mathbb{R}$ is a constant, to linear order in λ, neglecting terms of $\mathcal{O}(\lambda^2)$. Represent each term in your results graphically, by assigning to each index i a node with label i and by connecting node i with node j if Δ_{ij} appears in the term under consideration.