Multivariate Calculus

(Multiple Integrals and Vector Calculus)

Problem Sheet 2: "Derivative operators grad, div, curl, and related topics"

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Question 1. (Useful formulae for derivative operators)

For vector fields $A, B : \mathbb{R}^3 \to \mathbb{R}^3$ show the following identities.

a) $\nabla(A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla A) + A \times (\nabla \times B) + B \times (\nabla \times A)$

b) $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

c) $\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$

d) $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \Delta A$

Here, $\Delta = \text{div} \circ \text{grad}$ is the Laplacian.

Question 2. (Conservative vector fields)

Let $B : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ open, be a differentiable vector field.

a) What does it mean for B to be conservative? State necessary and sufficient condition for B to be conservative.

b) For a given $x \in U$ define a curve $X : [0,1] \to U$ by $X(t) = tx$. Show that a potential V for B is given by

$$V(x) = \int_{X([0,1])} B \cdot dx \; . \tag{1}$$

Which implicit assumption(s) have been made by writing down Eq. (1)?

c) For the domain $U = \mathbb{R}^n$ consider the vector fields given by (i) $B(x) = x$, (ii) $B(x) = r^p x$, (iii) $B(x) = (a \cdot x)b$ and, for the case $n = 3$, (iv) $B(x) = a \times x$, where a and b are fixed vectors, $p \geq 0$ and $r(x) = |x|$ is the radius function. Compute the curl of these vector fields.

d) For the conservative vectors fields from part c) find an associated potential.

Question 3. (Vector potential)

Let $B : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^3$ open, be a differentiable vector field in three dimensions

a) What is a vector potential for the vector field B? State necessary and sufficient conditions for a vector potential to exist.

b) Show that a vector potential for B is given by

$$A(x) = \int_0^1 dt\, t\, B(tx) \times x \; . \tag{2}$$

What are the implicit assumption in writing down Eq. (2).

c) Compute the divergence of the vector fields from Question 2, part c).

d) Write down a vector potential for the vector fields from Question 2, part c) where it exists. Also, write down a vector potential for a vector field $B(x) = b$, where $b \in \mathbb{R}^3$ is a constant vector.

Question 4. (Laplacian for radius-dependent functions)
On $\mathbb{R}^n$ with cooridnates $x = (x_1, \ldots, x_n)^T$ the Laplacian Δ is defined as

$$\Delta = \text{div} \circ \text{grad} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \, .$$

a) For a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ work out how the Laplacian acts on $f \circ r$, where $r(x) = |x|$ is the radius function.

b) Find the solutions to the Laplace equation $\Delta(f \circ r) = 0$.

c) Consider the Poisson equation $\Delta(f \circ r)(x) = \rho(x)$, where ρ is the function

$$\rho(x) = \begin{cases} \rho_0 & \text{for} \quad r(x) \leq R \\ 0 & \text{for} \quad r(x) > R \end{cases} \, ,$$

with $\rho_0, R \in \mathbb{R}$ constants. Find the solutions to this equation. Specialise to the solution with the following properties: (i) the solution and its first derivative are continuous at $r = R$ (ii) there are no singularities (iii) $f(r) \to 0$ as $r \to \infty$.

Question 5. (Vector potential and curl differential equation)
Consider $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)^T$, the cylindrical radius function $\rho(x) = \sqrt{x_1^2 + x_2^2}$ and vector fields $j : U \to \mathbb{R}^3$.

a) Assuming that a vector potential for a vector field $B : U \to \mathbb{R}^3$ exists, write the differential equation

$$\nabla \times B = j \tag{3}$$

for B in terms of this vector potential. Show that, for a suitable condition on the vector potential, this becomes a Poisson equation.

b) For a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ work out $\Delta(f \circ \rho)$, that is, the action of the Laplacian on functions which depend on the cylindrical radius only.

c) For $j = 0$, solve Eq. (3) by considering the associated Poisson equation for vector potentials A which only depend on the cylindrical radius. Also compute B.

d) Do the same as in Part c) but with

$$j(x) = \begin{cases} Je_3 & \text{for} \quad \rho(x) \leq R \\ 0 & \text{for} \quad \rho(x) > R \end{cases}$$

Compute the associated vector field B and ensure that it is nonsingular and continuous for $\rho(x) = R$.

Question 6. (Curve integrals)

Consider constant vectors $v_1, \ldots, v_k, v_{k+1} \in \mathbb{R}^n$ and define $u_a = v_{a+1} - v_a$ for $a = 1, \ldots, k$. The polygonal chain $C \subset \mathbb{R}^n$ with vertices $v_1, \ldots, v_k, v_{k+1}$ is defined as

$$C = \bigcup_{a=1}^{k} C_a \quad \text{where} \quad C_a = \{v_a + t\, u_a \,|\, t \in [0,1]\} \ .$$

a) Show that the curve integral of a conservative vector field (over any curve) only depends on the vector field's values at the curve's end points. Show that the integral of a conservative vector field over any closed curve vanishes.

b) Compute the curve integral $\int_C A \cdot dx$ for a constant vector field A, where C is a polygonal chain.

c) Consider a vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by $A(x) = Mx$, where M is an $n \times n$ matrix. Compute the curl of A and state under which conditions it vanishes. Compute the curve integral $\int_C A \cdot dx$ for a polygonal chain C and discuss the result.

Question 7. (Derivative operators and commutators)

We would like to think of differential operators as linear maps which act on infinitely many times differentiable functions f. By a zeroth order differential operator we simply mean a given function a which acts by multiplication, $f \mapsto af$. A first order derivative operator is of the form $O = a + \sum_i a_i \partial_i$, where the a, a_i are given functions and it acts as $f \mapsto af + \sum_i a_i \partial_i f$ and similarly for higher-order differential operators. The commutator of two differential operators P, Q is a differential operators defined as $[P, Q](f) = P(Q(f)) - Q(P(f))$ or, omitting the argument, by $[P, Q] = P \circ Q - Q \circ P$. Obviously, the commutator is anti-symmetric, $[P, Q] = -[Q, P]$.

a) For three differential operators P, Q and R show that $[P \circ Q, R] = P \circ [Q, R] + [P, R] \circ Q$ and $[P, Q \circ R] = Q \circ [P, R] + [P, Q] \circ R$.

b) Show that $[\partial_i, \partial_j] = [x_i, x_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$.

c) In three dimensions, define the first order differential operators $J_i = \epsilon_{ijk} x_j \partial_k$. Work out the commutators $[J_i, x_j]$, $[J_i, \partial_j]$ and $[J_i, J_j]$.

d) Show that $[J_i, \Delta] = 0$, where $\Delta = \sum_i \partial_i^2$ is the Laplacian.