Multivariate Calculus

(Multiple Integrals and Vector Calculus)

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Problem Sheet 1: "Partial and total derivatives"

Question 1. (Gradient)

The gradient, ∇f, of a differentiable function $f : U \to \mathbb{R}$, with $U \subset \mathbb{R}^n$ open, is defined by $\nabla f(x) := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)(x)$. Tackle the following problems which involve the gradient.

a) For two differentiable functions $f, g : U \to \mathbb{R}$ show that $\nabla(fg)(x) = g(x)\nabla f(x) + f(x)\nabla g(x)$, where fg is the product function, defined by $fg(x) = f(x)g(x)$.

b) For two differentiable functions $f, g : U \to \mathbb{R}$ and g non-vanishing, find a formula for $\nabla\left(\frac{f}{g}\right)$.

c) For two functions $f, g : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2z + xy^3$ and $g(x, y, z) = 2z^2y - x^2y^2$, work out $\nabla(fg)(p)$ and $\nabla\left(\frac{f}{g}\right)(p)$, where $p = (-1, -1, 1)^T$.

d) Consider a differentiable matrix-valued function M which depends on variables $x = (x_1, \ldots, x_n)^T$ such that $M(x)$ are invertible $n \times n$ matrices. Show that $\nabla\det(M) = \det(M)\mathrm{tr}(M^{-1}\nabla M)$. (Here, ∇M means that the gradient is applied to every entry of M.)

Question 2. (Gradient of radius-dependent functions)

Work out the gradient for the following functions which depend on a radius.

a) Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and the 'radius function' r given by $r(x) = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Work out the gradient of the function $\tilde{f} = f \circ r$.

b) Apply the result from part a) to the functions (i) $\tilde{f}(x) = r(x)^{-p}$ for $p \in \mathbb{R}$ and (ii) $\tilde{f}(x) = \ln(r(x))$.

c) Consider a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with coordinates $x = (x_1, x_2)^T$ on $\mathbb{R}^2$ and the radius function $\rho(x) = |x|$. Find the gradient of the function $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ defined by $\tilde{f}(x, z) = f(\rho(x), z)$.

d) Apply the results from part c) to the function (i) $\tilde{f}(x, z) = \rho(x)^{-p} + cz$ and (ii) $\tilde{f}(x, z) = \ln(\rho(x)) + cz$, where $p, c \in \mathbb{R}$.

Question 3. (Jacobi matrix and fix points)

For a differentiable function $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ open, consider the system of first order, ordinary differential equations given by

$$\frac{dx}{dt}(t) = f(x(t)) \,. \tag{1}$$

a) A point $\xi \in U$ is called a fixed point of the differential equation (1) if $f(\xi) = 0$. Discuss the significance of a fixed point for the differential equation (1).

b) For a fixed point ξ of the differential equation (1) write $x(t) = \xi + \epsilon(t)$ and find a linear differential equation for ϵ which approximates Eq. (1), neglecting terms of $\mathcal{O}(|\epsilon|^2)$.

c) Solve the linearised system from part b) to find out about the solution to Eq. (1) near a fixed point ξ, assuming the Jacobi matrix $Df(\xi)$ can be diagonalised. Discuss your results.

d) Consider a two-dimensional example with a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (-1 - y - x^3 - xy^2, -x - x^2 y)^T$. Find the stationary points and discuss the qualitative behaviour of the solutions to the linearised system (1) near the fix points, using the earlier results.

Question 4. (Cylindrical coordinates)

Consider $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)^T$ and the map to cylindrical coordinates (r, φ, z) defined by $x^T = (r\cos(\varphi), r\sin(\varphi), z)$.

a) Compute the Jacobi matrices of this map and the inverse map and work out the standard basis vectors in cylindrical coordinates defined by

$$e_r = \frac{\partial x}{\partial r} , \qquad e_\varphi = \frac{1}{r}\frac{\partial x}{\partial \varphi} , \qquad e_z = \frac{\partial x}{\partial z} .$$

b) Write the standard unit vectors e_1, e_2 and e_3 in $\mathbb{R}^3$ in terms of the cylindrical standard basis vectors from part a).

c) Work out the gradient in cylindrical coordinates. (This involves expressing the gradient in terms of partial derivatives with respect to cylindrical coordinates and writing the result as a linear combination of the cylindrical standard basis vectors.)

Question 5. (Homogenous functions)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $k \in \mathbb{Z}$ if $f(\lambda x) = \lambda^k f(x)$ for all $x \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$.

a) Show that the partial derivatives of a (differentiable) homogeneous function of degree k are homogeneous functions of degree $k - 1$.

b) Show that a (twice differentiable) homogeneous function of degree k satisfies

$$\nabla f(x)\, x = k\, f(x) , \qquad x^T H f(x)\, x = k(k-1) f(x) , \tag{2}$$

where $\nabla f(x)$ is the gradient and $Hf(x)$ is the Hesse matrix.

c) Which conclusions can you draw from the results in part b) about the stationary points of a homogeneous function?

d) Show that homogeneous functions $f : \mathbb{R} \to \mathbb{R}$ of degree k in a single variable are of the form $f(x) = c\, x^k$, where $c \in \mathbb{R}$ is a constant. Show that the class of homogeneous functions in multiple variables is richer by presenting some suitable examples of homogeneous functions in two variables.

Question 6. (Stationary points for functions with multiple variables)
Compute the gradient and the Hesse matrix for the following functions, find their stationary points and determine their nature (maximum, minimum or saddle).

a) $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 + z^2 + xz^2$.

b) $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x^T Mx - b \cdot x + c$, where M is a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ a vector and $c \in \mathbb{R}$.

c) $f : (0, \pi)^3 \to \mathbb{R}$ given by $f(x, y, z) = \sin(x + y) + \sin(x + z) + \sin(y + z)$.

Question 7. (Potentials from gradients)
For a three times differentiable function $W : \mathbb{R}^n \to \mathbb{R}$, define an associated function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \frac{1}{2}|\nabla W(x)|^2$.

a) Show that any stationary point x of W (with a non-singular Hesse matrix $HW(x)$) is a minimum of f with $f(x) = 0$.

b) Within the framework of part a), consider $W = ax + by + e^{-x} + e^{-y}$, where $a, b > 0$ are constants, and compute the associated function f. Use the result from part a) to find a minimum of f.

Question 8. (Invariant functions)
Consider a map $G : U \to \mathbb{R}^{n^2}$ with $t = (t_1, \ldots, t_k)^T \mapsto G(t)$ which provides a family of invertible $n \times n$ matrices $G(t)$, such that $G(0) = \mathbb{1}_n$. (Think of rotation matrices as a function of angles, for example.) A Taylor expansion around $t = 0$ gives $G(t) = \mathbb{1}_n + \sum_{a=1}^{k} t_a T_a + \mathcal{O}(t^2)$, where $T_a = \frac{\partial G}{\partial t_a}(0)$ are $n \times n$ matrices called the generators. We also define the vector space of matrices $\mathcal{L} = \text{span}(T_1, \ldots, T_k)$ spanned by the generators. We say that a function $V : \mathbb{R}^n \to \mathbb{R}$ is invariant under a family of matrices G if $V(x) = V(G(t)x)$ for all x and t.

a) Assume all matrices $G(t)$ are rotation matrices. Show that in this case the associated vector space $\mathcal{L}$ consists of anti-symmetric matrices. Show if $G(t)$ gives the set of all rotation matrices (in n dimensions) then $\mathcal{L}$ is the vector space of all anti-symmetric $n \times n$ matrices.

b) Assume that x is a stationary point of a function V which is invariant under a matrix family G. Show that $G(t)x$ is also a stationary point.

c) Consider a function V invariant under a matrix family G with associated vector space $\mathcal{L}$. Show that a stationary point x of V satisfies $HV(x)\, Tx = 0$ for all $T \in \mathcal{L}$. (Here, HV is the Hesse matrix of V.) What does this result tell us about the nature of the stationary point?

d) Show that the function $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = \frac{1}{4}(|x|^2 - v^2)^2$ (where $v \in \mathbb{R}$ is a non-zero constant), is invariant under n-dimensional rotations and verify the statements from parts b) and c) for this example.