A14119H1

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SHORT OPTIONS

TRINITY TERM 2025

Tuesday, 17 June
2.30 pm to 4.00 pm

Answer **two** *questions from the option for which you have entered.*

Start the answer to each question in a fresh book.

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

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Some sections may start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. (a) Show that an analytic function satisfies the Cauchy-Riemann conditions. [3]

(b) Consider a complex function, $f(z) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$ such that

$$\begin{aligned} u &= \frac{x\cos(4xy) - y\sin(4xy)}{x^2 + y^2}\mathrm{e}^{-2(x^2-y^2)}, \\ v &= -\frac{x\sin(4xy) + y\cos(4xy)}{x^2 + y^2}\mathrm{e}^{-2(x^2-y^2)}. \end{aligned}$$

Show that f is analytic. Express f as a function of $z = x + \mathrm{i}y$. [5]

(c) Find the residue of

$$f(z) = \frac{\mathrm{cotan}z\ \mathrm{coth}z}{z^3}$$

at $z = 0$. [5]

(d) Consider the function

$$f(z) = \frac{1}{(z^2 + a^2)(z^2 + b^2)},$$

where a and b are real and $b \geq a > 1$. Find the singularities and corresponding residues. [6]

(e) Choose a contour integral to calculate

$$F(a, b) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)}.$$

[6]

2.

(a) State Cauchy's Residue Theorem. Assume that any complex function has a Laurent expansion around a singularity and use that to prove the theorem. [3]

(b) Find the Laurent series of the function

$$f(z) = \frac{1}{z^4 - 1}$$

expanded around $z = 0$, for (i) $|z| < 1$ and (ii) $|z| > 1$. [5]

(c) Describe the singularities of

$$f(z) = \frac{z^\alpha}{z^2 + 2z\cos(\pi\gamma) + 1},$$

where α and γ are real and $|\alpha| < 1$ and $|\gamma| < 1$. Find the corresponding residues. Define three different sets of branch cuts for this function. [5]

(d) Pick a suitable contour and use contour integration to calculate

$$F(\alpha, \gamma) = \int_0^\infty \frac{x^\alpha \mathrm{d}x}{x^2 + 2x\cos(\pi\gamma) + 1}.$$

Show that $F(\alpha, \gamma) = F(\gamma, \alpha)$. [7]

(e) Use the method of contour integration to find

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(2 + \cos\theta)^2}$$

[5]

3.

(a) State Cauchy's integral formula, prove it and show how it can be used to construct the Taylor series of an analytic function. [3]

(b) Consider a square contour, C_N with vertices at

$$\left(N+\frac{1}{2}\right)(1+\mathrm{i}),\quad \left(N+\frac{1}{2}\right)(-1+\mathrm{i}),\quad \left(N+\frac{1}{2}\right)(-1-\mathrm{i}),\quad \left(N+\frac{1}{2}\right)(1-\mathrm{i}).$$

Consider a function $g(z)$ with a finite number, M, of poles. Let $|g(z)| \leq \alpha/|z|^\beta$ on C_N where α and $\beta > 1$ are constants independent of N. Use a contour integral, on the contour C_N, to show that

$$\sum_{n=-\infty}^{\infty} g(n) = -\sum_{k=1}^{M} \mathrm{Res}[\pi \mathrm{cotan}(\pi z) g(z), z_k],$$

where z_k, with $k = 1, \cdots, M$ are the poles of $g(z)$. You may assume that $\mathrm{cotan}(\pi z) < A$ on C_N where A is a constant independent of N and that none of the poles, z_k are integers. [6]

(c) Consider the function

$$f(z) = \sqrt{2} z \mathrm{e}^{i\pi/4} + (1 - 2i).$$

Show that $w = f(z)$ is a conformal transformation. Sketch what the rectangle, with vertices at

$$0,\quad 2,\quad 2+\mathrm{i},\quad \mathrm{i},$$

is mapped onto via this transformation. [5]

(d) Consider the function

$$h(z) = \frac{z^2}{1+\mathrm{e}^z}$$

Determine the poles of this function and the corresponding residues. [5]

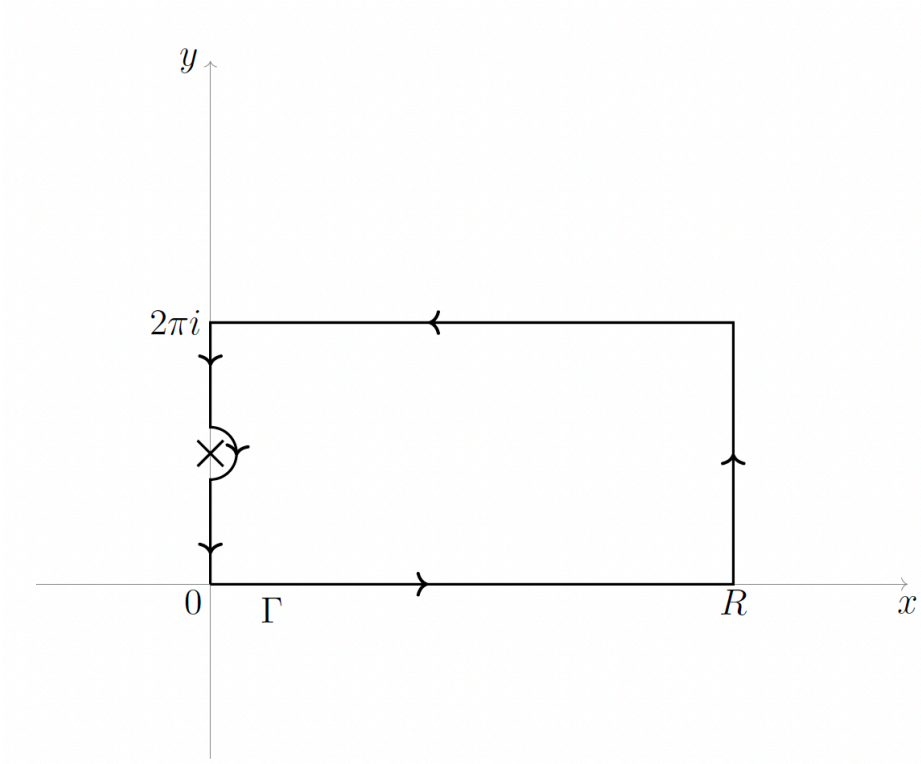
(e) Consider the contour, Γ_R in figure below. Use the contour integral

$$\oint_{\Gamma} h(z)\mathrm{d}z$$

to calculate

$$\int_0^{\infty} \frac{x\mathrm{d}x}{1+\mathrm{e}^x}$$

[6]



Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. a) A dwarf star of mass m has a gas giant planetary companion of mass αm, where $0 < \alpha \leq 1$. The two are separated by a distance a and are on circular orbits. Show that such a system is gravitationally bound. [9]

b) The dwarf star in the system above suffers spherically-symmetric mass loss, such that it retains only a fraction β of its original mass after the loss, where $0 < \beta \leq 1$. Derive a condition on β for the system to remain gravitationally bound. [7]

c) State the criteria for the currently-accepted definition of a planet. How does a dwarf planet differ from these criteria? [5]

d) State the strengths and weaknesses of each of the microlensing and the direct imaging techniques used to detect exoplanets. [4]

2. A spherically-symmetric star has a radial profile of the mass volume density ρ given by a power-law, such that $\rho(r) \propto r^{-n}$, where r is the radius and n the (constant) power-law index ($0 < n < 1$).

a) Use the equations of stellar structure to show that the mass M enclosed within a radius r is a power-law of the radius and state the power-law index. Similarly, show that the pressure P at a radius r is also a power-law of the radius and state the power-law index. [4]

b) The stellar radiative transfer equation is given by

$$L(r) = -4\pi r^2 \, \frac{4ac}{3\kappa(r)\rho(r)} \, T^3(r) \, \frac{dT(r)}{dr} \ ,$$

where L is the stellar luminosity, a a constant, c the speed of light, κ the opacity, and T the temperature.

i) In low-mass stars, the opacity is given by $\kappa(r) \propto \rho(r) T^{-3}(r)$ and the pressure is dominated by the gas pressure $P_{\rm gas}(r) = \dfrac{\rho(r)}{\mu m_{\rm H}} kT(r)$, where μ is the mean molecular mass (that can be considered constant), $m_{\rm H}$ the mass of the Hydrogen atom, and k Boltzmann's constant. Show that the temperature T at a radius r is a power-law of the radius and state the power-law index. In turn, show that the luminosity L at a radius r is also a power-law of the radius and state the power-law index. Hence derive the luminosity – mass relation of low-mass stars. [5]

ii) In high-mass stars, the opacity $\kappa = \kappa_{\rm th}$ = constant and the pressure is dominated by the radiation pressure $P_{\rm rad} = \dfrac{1}{3} a T^4(r)$. Show that the temperature T at a radius r is a power-law of the radius and state the power-law index. In turn, show that the luminosity L at a radius r is also a power-law of the radius and state the power-law index. Hence derive the luminosity – mass relation of high-mass stars. [5]

c) State the main chain of fusion reactions that leads to energy production in the Sun, and state the net reaction. [5]

d) Given the net fusion reaction above, and that only 10% of the masses of low-mass stars on the main sequence burn through this reaction, calculate the main sequence lifetime of a star of i) 1 solar mass and ii) 2 solar masses. You can assume the stellar luminosities to be constant during these main sequence lifetimes. [6]

[Take the total mass of the Sun to be $\rm M_\odot = 2.0 \times 10^{30}$ kg, the total luminosity of the Sun to be $\rm L_\odot = 4.0 \times 10^{26}$ W and the speed of light c to be 3.0×10^8 m s^{-1}. The mass of a Hydrogen atom is 1.6735×10^{-27} kg, the mass of a Helium atom is 6.6460×10^{-27} kg.]

3. a) A hypothetical force leads to the following term in the Friedmann equation:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{F}{R} \ ,$$

where R is the scale factor of the universe, $\dot{R}$ its derivative with respect to time, and F a constant. In a universe dominated by this force, determine the dependence of the scale factor R on time. What are the conditions on F to have i) a universe with an accelerating expansion and ii) a universe with a big crunch? What is the present age of the universe given a Hubble constant today of $H_0 = 70$ km s^{-1} Mpc^{-1}? [8]

b) A universe is dominated by the cosmological constant Λ, such that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3}$$

at all times. In this universe, determine the dependence of the scale factor R on time. The particle horizon is defined as

$$r_{\rm p}(t) \equiv R(t) \int_0^t \frac{c\,dt'}{R(t')} \ ,$$

where c is the speed of light and t the time. Calculate $r_{\rm p}(t)$, sketch its dependence on time, and evaluate it today (time $t = t_0$) and at decoupling (when the scale factor was $R = R_{\rm d}$). Hence determine the angle subtended today by the particle horizon at decoupling. Show that for $t_0 \gg (\Lambda/3)^{-1/2}$ this angle is necessarily equal to 1 radian. [11]

c) State the basic tenets of the cosmological principle. How has the historical evolution of our models of the universe embodied these tenets? How does the large-scale structure of our universe agree (or not) with those tenets? [6]

Section S3 QUANTUM IDEAS

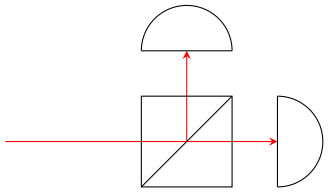
1. (a) The electric field in a classical light wave propagating along the z axis is

$$\vec{E}(z,t) = \mathrm{Re}[(A_H\hat{i} + A_V\hat{j})e^{i(kz-\omega t)}], \quad (1)$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors along the x and y axes, ω is the angular frequency, k is the wavevector, t is time, and A_H and A_V are the complex amplitudes corresponding to the horizontal and vertical polarisations.

(i) Write down the normalised quantum state of the photons comprising this beam in the canonical basis $\{|H\rangle, |V\rangle\}$, representing, respectively, the horizontally and vertically polarised photons. [1]

(ii) The polarisation of the field is measured using a polarising beam splitter and two detectors, as shown below.



Explain how the photocurrents measured by the detectors are consistent with the photon detection statistics observed for the polarisation measurement in the quantum setting. [5]

(b) Measurements on multiple photons, each prepared in a pure state $|\psi\rangle$, detect horizontal polarisation in 3/4 of the events, and vertical in 1/4 of the events. What can be deduced about $|\psi\rangle$ from these results? [2]

(c) Subsequent measurements on multiple copies of the same state, performed in the diagonal polarisation basis $|\pm 45^\circ\rangle = \dfrac{|H\rangle \pm |V\rangle}{\sqrt{2}}$, showed the two results to occur with equal probability.

(i) Sketch an experimental scheme implementing this measurement. [1]

(ii) What can be said about $|\psi\rangle$ based on these additional results? [5]

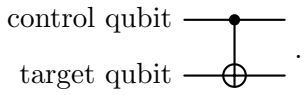
(iii) Sketch the polarisation patterns for all possible answers to part (ii). [3]

(iv) Propose an additional measurement that can help uniquely identify $|\psi\rangle$. Determine the outcome probabilities of this measurement for each solution of part (ii). [3]

(d) (i) Explain the concept of the mixed state. Give an example of a physical situation in which a mixed state emerges. Are mixed states elements of the Hilbert space? [2]

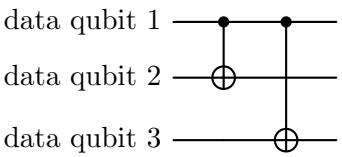
(ii) Measurements of a polarization state in the canonical, diagonal and circular bases all yielded each outcome occurring with a probability 1/2. What is this state? Prove your answer by calculating the probabilities. [3]

2. The Conditional NOT (C-NOT) gate is a common tool in quantum computation, which flips the value of the second (target) qubit if the first (control) qubit value is 1. The notation for the C-NOT gate in circuit diagrams is as follows:



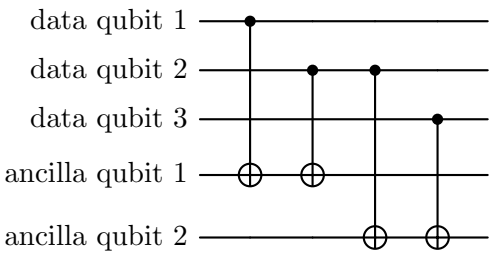
(a) Write down the matrix of the C-NOT gate operator $\hat{A}$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. [2]

(b) Show that the circuit shown below transforms three data qubits, initially in the state $(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle$, into $|\Psi\rangle = \alpha|000\rangle + \beta|111\rangle$. [3]



(c) Such *redundant encoding* of a qubit is a basis of error correction in a quantum computer. We consider *bit-flip errors*, which are associated with the Pauli operator $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ occurring in one of the qubits. Write down the states that $|\Psi\rangle$ transforms into after the bit-flip error occurs in (i) qubit 1, (ii) qubit 2, (iii) qubit 3. [3]

(d) To detect the error, a pair of ancillary qubits, initially in state $|0\rangle$, is introduced and the following set of operations is performed:



For each of the cases listed in part (c), as well as for the case that no error has occurred, write down the resulting 5-qubit state. [5]

(e) The two ancillary qubits are subsequently measured in the computational basis. Show that this measurement allows the observer to unambiguously identify the qubit in which the bit-flip error has occurred, or that no error has occurred. For each of these four cases, find the state of the three data qubits after the measurement on the ancillary qubits. Show that state $|\Psi\rangle$ can be recovered from that state by means of a unitary operation, and write down that operation. [5]

(f) Find the resulting state of the three data qubits if the above-described error correction protocol has been applied after the bit-flip error has occurred in both qubits 1 and 2. Will the error correction protocol work in this case? [3]

(g) Consider the case of a 'partial' error, associated with the operator $a\hat{\sigma}_x + b\hat{\mathbf{1}}$ (where $\hat{\mathbf{1}}$ is the identity operator and $|a|^2 + |b|^2 = 1$) applied to one of the qubits. Will the above-described error correction protocol be able to correct this error? Justify your answer. [4]

3. The spin states of an electron comprise a two-dimensional Hilbert space with the canonical basis consisting of the states $|\uparrow\rangle$ and $|\downarrow\rangle$ ('spin-up' and 'spin-down', respectively).

(a) Application of a magnetic field to the electron transforms the canonical basis states into $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$, respectively. Find the matrix of the operator $\hat{A}$ defining this transformation. Is this operator Hermitian? Is it unitary? [3]

(b) When two spins are interacting, the energy of the state $|\downarrow\downarrow\rangle$ becomes $\hbar\omega_0$, whereas the energies of the states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ remain zero. Find in the canonical basis

(i) the Hamiltonian defined by this interaction, [2]

(ii) the matrix of the evolution operator $\hat{U}$ arising from this interaction being active for time $t = \pi/\omega$. [2]

(c) Two spins, initially prepared in states $(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$ and $|\uparrow\rangle$, are subjected to a sequence of operations as described below. Find their state after each of these operations:

(i) the operation described in part (a) applied to the first spin only; [2]

(ii) the interaction described in part (b); [1]

(iii) step (i) repeated. [3]

(d) Repeat part (c) for the second qubit initially in the $|\downarrow\rangle$ state. [6]

(e) Based on your answer, write down the matrix of the operator corresponding to the sequence of operations described in part (c). [2]

(i) What is the name of this operation in the context of quantum computing? What about each individual element of the sequence? [2]

(ii) Can this operation lead to entanglement of the first and second qubit if they were initially unentangled? Present an example or argument in support of your answer. [2]