A10289H1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Monday 1st September 2025, 09:30 am – 12 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) Evaluate the limit

$$\lim_{x\to 0}\left[\frac{\left(1-e^{2x}\right)^2}{1-\cos x}\right].$$

[3]

(b) Find a solution in terms of a trigonometric function for the integral

$$\int_0^x \frac{\mathrm{d}t}{1+t^2},$$

and use your result to deduce the first four non-zero terms of the Taylor series expansion of $\arctan(x)$ about $x = 0$. [4]

2. A central force field in three-dimensional space may be written as

$$\mathbf{F}(r) = \mathbf{r}f(r),$$

where $\mathbf{r}$ is a radial vector and where $f(r)$ is a scalar function only of radius r.
(a) Obtain an expression for $\nabla \cdot \mathbf{F}$ in terms of $f(r)$ and its derivative $\mathrm{d}f/\mathrm{d}r$.
(b) Hence calculate $\nabla \cdot \mathbf{F}$ for the case $f(r) = r^{n-1}$, with n an integer, and discuss what happens to the divergence for the case $n = -2$. [6]

3. A fly is moving along the curve parameterised by $x = \cos(t)$, $z = \sin(2t)$ in a vertical plane above a heat source. The temperature distribution of the air in this plane is given by:

$$T(x, z) = 20e^{-x^2} - (z+1)^3.$$

Find the rate of change of temperature that the fly experiences as a function of t. [5]

4. For the vector field $\mathbf{F} = (-x, y, z)^T$, with (x, y, z) Cartesian coordinates, evaluate $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the surface of the plane $x + y + 2z = 4$ in the octant where x, y and z are all positive. [5]

5. State Stokes' Theorem for an arbitrary vector field $\mathbf{A}$. Now consider the vector field

$$\mathbf{A}(x, y) = -\frac{y}{2}\nabla x + \frac{x}{2}\nabla y,$$

where (x, y) are Cartesian coordinates. Use Stokes' Theorem to calculate the area of the region enclosed by the curve C, which is parameterised by $x = \sin(2t)$ and $y = \sin(t)$, with t a parameter in the range $0 \le t \le \pi$. [7]

6. Using the change of variables $u = x - 4y$ and $v = 3x + y$, evaluate

$$\iint_D \frac{x-4y}{3x+y}\mathrm{d}x\mathrm{d}y,$$

where D is the parallelogram bounded by the lines $y = 1 - 3x$, $y = 3(1 - x)$, $y = x/4$ and $y = (x - 4)/4$. [5]

7. A pulse propagating from the positive side ($x \geq 0$) towards the fixed end ($x = 0$) of a semi-infinite string under constant tension has the initial triangular shape:

$$y(x, t = 0) = \begin{cases} 0, & x > 2b \\ -2a + ax/b, & b \leq x \leq 2b \\ -ax/b, & 0 \leq x < b. \end{cases}$$

Sketch the shape of the string at $t_1 = b/(2c_s)$, $t_2 = 3b/(4c_s)$ and $t_3 = b/c_s$, where c_s denotes the speed of wave propagation in the string and a and b are real constants. [5]

Section B

8. (a) Show that the equation

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d$$

represents a plane, where $\mathbf{r}$ is the position vector from the origin to a point on the plane, $\hat{\mathbf{n}}$ is a unit vector normal to the plane, and $d > 0$ is a real constant. [3]

(b) Write down the equations of the planes in terms of the Cartesian coordinates (x, y, z) for the following cases:

(i) The plane passes through the points $(3, 1, -2)$, $(1, -1, 2)$ and $(-1, -2, 0)$. [4]

(ii) The plane is tangent to the paraboloid $z = 4 - (x^2 + y^2)$ at the point $(1, 1, 5)$. [4]

(c) Find the length of the major axis of the ellipse which is formed when the cylinder $x^2 + y^2 = R^2$ intersects the plane $2x + y - 4z = 3$, where R is a real constant. [5]

(d) Find the volume above the x-y plane that is bounded by the cylinder $x^2 + y^2 = 9$ and the paraboloid $3z = x^2 + y^2$. [4]

9. Consider the function of two variables

$$z(x, y) = f(x)g(x, y),$$

where $f(x) = \sin(x)$ and $g(x, y) = e^{-\left(x^2+y^2\right)}$.

(a) Show that there are multiple stationary points of z, that these stationary points lie along the x-axis and that their positions satisfy an equation of the form $\tan(x) = k(x)$. Derive the expression for $k(x)$. [5]

(b) Write down to second order the general expression for a Taylor series expansion of a function of two variables. Expand the function $z(x, y)$ from part (a) as a Taylor series in (x, y) to second order about a general stationary point, writing the value of z at a stationary point $(x_s, 0)$ as z_s. [3]

(c) State the general criteria for determining the nature of the stationary values of functions of two variables. [4]

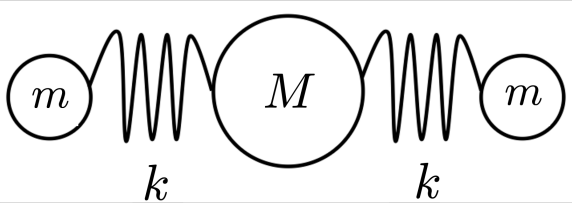
(d) Use those criteria to evaluate the nature of all the stationary points of the function $z(x, y)$. [4]

(e) Now consider the function

$$w(x, y) = h(x)g(x, y),$$

where $h(x) = x^m$, $m > 0$ is an integer and $g(x, y)$ is as given above. Show that w has a stationary value at the origin whose nature cannot be determined by considering derivatives only up to second order. [4]

10. A simple model for a linear triatomic molecule is three atoms connected by two springs, as shown below. Assume that the atoms are constrained to move in one dimension, and that the springs are identical with spring constant k. Also assume that the central atom, with mass M, is more massive than the other two, which we assume to have equal masses, m. Considering the atoms in the triatomic molecule from left to right, define their displacements from their equilibrium positions as x_1, x_2, x_3.



(a) Find an equation of motion for each of the three masses. [3]

(b) Explain in words what is meant by a normal mode of a system. How many normal modes exist for the linear triatomic molecule? [2]

(c) Obtain expressions for the frequencies of the normal modes of the simple model of the triatomic molecule described above. [5]

(d) Find the relative motions and sketch the displacements as a function of time of each of the three masses in each of the normal modes. [4]

(e) Now consider the case where $M = 4m$ and the atoms are released from rest with initial displacements from their equilibrium positions of $x_1 = d$, $x_2 = -d$ and $x_3 = 0$. Find expressions for the subsequent displacement of each of the three masses as a function of time. [6]

11. The field $\phi(x,t)$ satisfies the equation

$$\frac{\partial^3\phi}{\partial t^3} + \mathrm{i}\omega_n \frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^3\phi}{\partial t \partial x^2} - \mathrm{i}\omega_d c^2 \frac{\partial^2\phi}{\partial x^2} = 0, \tag{1}$$

where c, ω_n and ω_d are positive constants.

(a) Show that this equation has wave solutions of the form $\phi(x,t) \propto e^{\mathrm{i}(kx-\omega t)}$, if ω and k satisfy the dispersion relation

$$k^2 = \frac{\omega^2(\omega-\omega_n)}{c^2(\omega-\omega_d)}.$$ [3]

(b) Assuming that $\omega_n > \omega_d$, determine the values of ω for which the waves cannot propagate. [3]

(c) Find the phase and group velocities in the limits

(i) $|\omega| \to 0$,

(ii) $|\omega| \to \infty$,

(iii) $\omega \to \omega_n$,

(iv) $\omega \to \omega_d$. [8]

(d) Consider the case with $\omega_d = 0$ and $\omega_n = 3k_0c/2$, where k_0 is a constant. Using a solution of the form $\phi(x,t) = T(t)e^{\mathrm{i}k_0x}$, solve equation (**??**) with the initial conditions

$$\begin{aligned} \phi(x,0) &= 0, \\ \frac{\partial\phi}{\partial t}(x,0) &= 0, \\ \frac{\partial^2\phi}{\partial t^2}(x,0) &= 5k_0^2c^2e^{\mathrm{i}k_0x}. \end{aligned}$$

[6]