A10289H1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 12 June 2025, 9.30 am – 12.00 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Evaluate the following expressions:

$$\lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$$

and

$$\lim_{x\to 1} \frac{\ln(x)+1-x}{(e^x-e)^2}.$$

[4]

2. Two monochromatic plane waves, labelled by the subscripts + and −, respectively, have equal amplitudes and have frequencies $\omega_\pm = \omega_0 \pm \delta\omega$ and wave numbers $k_\pm = k_0 \pm \delta k$, where $\delta\omega \ll \omega_0$ and $\delta k \ll k_0$.

(a) By considering the superposition of these waves, derive an expression for the group velocity of the resulting wave packet and sketch a snapshot of the superposed waves. Be sure to label any special wavelengths on your sketch.

(b) Assuming the waves are propagating in a non-dispersive medium, express $\delta\omega$ in terms of ω_0, k_0 and δk. [6]

3. Suppose that $u = u(s,t)$ and $v = v(s,t)$.

(a) By calculating the total differentials of u and v, obtain expressions for $\mathrm{d}s$ and $\mathrm{d}t$ in terms of $\mathrm{d}u$, $\mathrm{d}v$ and partial derivatives of u and v with respect to s and t.

(b) By equating your expression for $\mathrm{d}s$ with the differential of $s(u,v)$, derive an expression for $(\partial s/\partial v)_u$ in terms of partial derivatives of u and v with respect to s and t. [5]

4. The joint probability density function $f_{XY}(x,y)$ over the continuous random variables X and Y is defined such that

$$f_{XY}(x,y) = 5\left(x^2 - y\right)$$

for $(-1 \le x \le 1,\ 0 \le y \le x^2)$ and zero otherwise. Calculate the following quantities:

(a) the marginal probability density function $f_X(x)$;

(b) the marginal probability density function $f_Y(y)$; and

(c) the expectation value for x^2. [7]

5. Evaluate the integral

$$\int_\Omega (3-z)x^2 \mathrm{d}S,$$

where Ω is the part of the conical surface $3 - z = \sqrt{x^2+y^2}$ that lies between $z = 1$ and $z = 2$, and dS is the element of area on Ω. [7]

6. The force acting on a particle in $\mathbb{R}^3$ is $\mathbf{F} = C_1\left(xyz\hat{\mathbf{x}} + x^2y\hat{\mathbf{y}} + x^2z\hat{\mathbf{z}}\right) - (C_2/r^3)\hat{\mathbf{r}}$, where (x,y,z) are Cartesian coordinates and $r = \sqrt{x^2+y^2+z^2}$ is the spherical radial coordinate. Calculate the work done by the force in moving the particle along the straight-line path between points A and B, with $A = (1,2,2)$ and $B = (2,4,4)$. [6]

7. In a simple model for one-dimensional gas dynamics, the density of particles ρ at any space-time location (x, t) is given by

$$\rho(x,t) = \frac{\rho(x+\lambda, t-\tau)}{2} + \frac{\rho(x-\lambda, t-\tau)}{2},$$

with $\lambda > 0$ and $\tau > 0$ real constants. You may assume that ρ varies on space and time scales that are much larger than λ and τ, respectively.

(a) Taylor expand this expression about (x, t) to the lowest order in λ and τ required to obtain a partial differential equation relating the time derivative of ρ to its spatial derivatives.

(b) Now suppose that (x_0, t_0) is a critical point for ρ. Given the form of the differential equation from (a), what, if anything, can be determined about the nature of this critical point? [5]

Section B

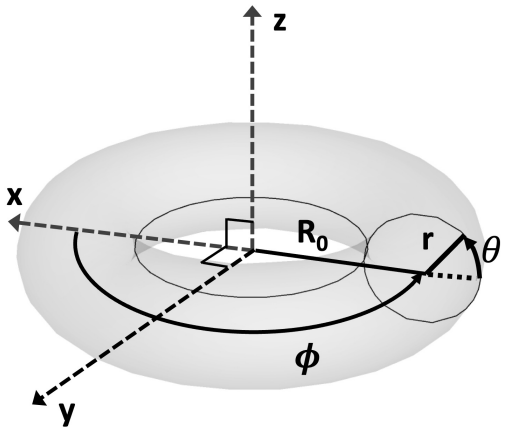
8. Toroidal coordinates (r, ϕ, θ) are related to Cartesian coordinates (x, y, z) via

$$x = R\cos\phi, \quad y = R\sin\phi, \quad z = r\sin\theta,$$

where

$$R = R_0 + r\cos\theta,$$

with $R_0 > 0$ a real constant. These coordinates are illustrated in the diagram below.



(a) Calculate the Jacobian determinant for the transformation from (x, y, z) to (r, ϕ, θ) coordinates. [5]

(b) Calculate the surface area and volume of a torus with $0 \leq r \leq a$, where a is a real constant. [5]

(c) State the divergence theorem, precisely defining all mathematical symbols used. [3]

(d) Verify the divergence theorem for the vector field $\mathbf{F} = (0,\ 0,\ z)^T$, defined over the part of the toroidal volume from (b) with $a/2 \leq r \leq a$ and $z \geq 0$. [7]

9. A light, uniform string of mass per unit length ρ is held under constant tension T. The transverse displacement of the string from its equilibrium position is y, the horizontal position along the string is x and time is t.

(a) Derive the wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0$$

that describes small-amplitude, transverse waves on the string. Clearly identify where the small-amplitude approximation is made and how c is related to the properties of the string. [4]

(b) A string of mass density ρ_1 extends from $x = -\infty$ to $x = 0$, where it is connected to another string of mass density ρ_2 that extends from $x = 0$ to $x = \infty$. A harmonic wave travels from $x < 0$ towards $x = 0$.

(i) Express the ratio k_1/k_2 of the wave numbers for waves propagating along each of the two segments of string in terms of ρ_1 and ρ_2. [2]

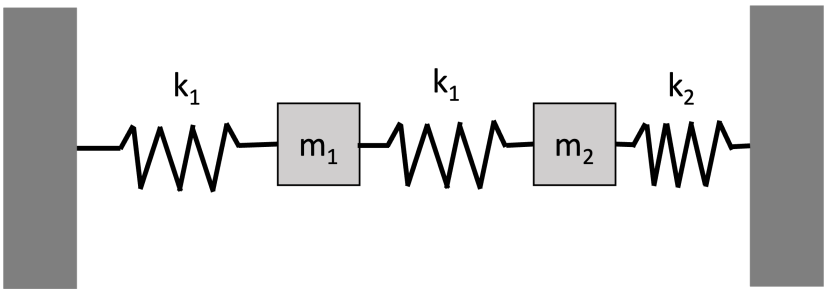
(ii) State and justify the boundary conditions that transverse oscillations along the two strings must satisfy. [3]

(iii) Calculate the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave. Express your answers in terms of ρ_1 and ρ_2. [3]

(iv) Calculate the ratio of reflected to incident power and the ratio of transmitted to incident power. Express your answers in terms of ρ_1 and ρ_2. [2]

(c) A point mass M is attached at the boundary between the two strings in part (b). Derive modified boundary conditions by considering the force on an infinitesimal segment of string spanning $x = 0$. Calculate the ratio of reflected to incident power and compare to your result from (b) in the limiting cases of $\omega \to 0$ and $\omega \to \infty$, with ω the angular frequency of the waves. [6]

10. Two masses m_1 and m_2 are connected to each other by a spring of spring constant k_1 and to adjacent walls by springs with spring constants k_1 and k_2, as shown in the diagram. Denote the displacements of the masses m_1 and m_2 from their equilibrium positions by x_1 and x_2, respectively.



(a) Write down the equations of motion for the masses. [4]

(b) Taking $m_2 = 2m_1$ and $k_2 = 3k_1$, calculate the frequencies associated with the normal modes of the system. [4]

(c) Calculate the ratio x_2/x_1 for each of the normal modes. [4]

(d) At $t = 0$ the masses are in their equilibrium positions with speeds

$$\frac{dx_1}{dt} = u, \quad \frac{dx_2}{dt} = 0.$$

Derive an expression for the position of the masses for $t > 0$. [4]

(e) Write down the normal coordinates for the system and express the kinetic energy in terms of the normal coordinates. What fraction of the total energy is associated with each normal mode? [4]

11. (a) Derive an expression for $\nabla \times (\phi \mathbf{A})$ in terms of ϕ, $\nabla\phi$, $\mathbf{A}$ and $\nabla \times \mathbf{A}$, where ϕ is a scalar and $\mathbf{A}$ is any vector in $\mathbb{R}^3$. [3]

(b) The vector $\mathbf{u} \in \mathbb{R}^3$ evolves in time t according to the equation

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{r}} = -\frac{\nabla p}{f(p)},$$

where $\mathbf{r}$ is the position vector, $p(\mathbf{r})$ is a scalar function of $\mathbf{r}$ and $f(p)$ is a scalar function of p. Derive an expression for $(\partial \boldsymbol{\omega}/\partial t)_{\mathbf{r}}$, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. [3]

(c) Given that $\mathbf{u}$ satisfies the differential equation from (b), calculate $(\partial \Gamma/\partial t)_{\mathbf{r}}$, where

$$\Gamma = \oint_C d\mathbf{r} \cdot (\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}),$$

with $\boldsymbol{\Omega} = \sin(t/t_0)\hat{\mathbf{z}}/t_0$, t_0 a real constant and C any static, closed curve in the $x-y$ plane enclosing area A. [8]

(d) Now consider the case where the curve C is a circle in the $x-y$ plane with radius $a(t) = a_0 t/t_0$ and $\mathbf{u}(\mathbf{r}, t = t_0) = (-x^2 y, xy^2, xyz)^T/(a_0^2 t_0)$. Calculate $(\partial \Gamma/\partial t)_{\mathbf{r}}$ for $t \geq t_0$. You should assume again that $\mathbf{u}$ satisfies the evolution equation from (b). [6]

You may find the following identity useful:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) + \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} + (\mathbf{A} \cdot \nabla)\,\mathbf{B},$$

where $\mathbf{A}$ and $\mathbf{B}$ are any two vectors in $\mathbb{R}^3$.