A10285H1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Tuesday 2nd September 2025, 09.30 – 12.00 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) Find the real and imaginary parts of each of the following expressions

$$\mathrm{e}^{-i\pi} \qquad \text{and} \qquad \frac{1+3i}{4-2i}.$$

(b) In an Argand diagram, sketch the locus of points that satisfy $|z| = 2\arg(z)$. [5]

2. A 2×2 Hermitian matrix H has non-degenerate eigenvalues λ_1 and λ_2 with corresponding normalised eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$.
(a) Find $\det(H)$ and $\mathrm{tr}(H)$ in terms of λ_1 and λ_2.
(b) Determine the values of $\mathrm{Im}\{2\lambda_1\lambda_2\}$ and $\mathbf{e}_1^\dagger(2\mathbf{e}_1 + \mathbf{e}_2)$, giving reasons for your answers. [5]

3. Find the perpendicular distance of the point $(1, 2, 2)$ from the straight line that passes through $(2, 3, 1)$ and $(3, 4, 2)$. [6]

4. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+x-2}{y+x}.$$

[5]

5. The vectors

$$\mathbf{A} = \mathbf{b} \times \mathbf{c}, \quad \mathbf{B} = \mathbf{c} \times \mathbf{a}, \quad \text{and} \quad \mathbf{C} = \mathbf{a} \times \mathbf{b},$$

are defined in terms of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$. Find the value of $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$, expressed in terms of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. [5]

6. Consider the 3×3 matrix

$$A = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of A and identify any degenerate eigenvalues.
(b) Find a corresponding set of three linearly independent eigenvectors, indicating clearly which eigenvector belongs to which eigenvalue. Verify that the chosen eigenvectors are indeed linearly independent.
(c) Use matrix row reduction to find the rank of the matrix $(A - \lambda I)$, where λ is the degenerate eigenvalue and I is the identity matrix. Explain why this result is expected, given the degeneracy of the matrix A. [7]

7. The displacement $y(t)$ of a particle satisfies the differential equation for a damped oscillator,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = 0,$$

for real constants γ and ω_0. Consider the case of very light damping with $\gamma \ll \omega_0$.

(a) Show that the displacement can be written as

$$y(t) \approx y_0 \mathrm{e}^{-\gamma t/2} \cos(\omega_0 t + \phi),$$

for constants y_0 and ϕ.

(b) Calculate the total energy of this oscillator, and show that the energy decreases exponentially with time.

(c) Hence show that the quality factor is given by $Q = \omega_0/\gamma$. [7]

Section B

8. (a) Express the set of simultaneous equations

$$\begin{aligned} x_1 + x_2 &= -3, \\ 2x_1 + x_2 + 3x_3 &= \ \ 4, \\ x_1 + 2x_2 + 3x_3 &= -1, \end{aligned}$$

in the form $A\mathbf{x} = \mathbf{b}$, where the matrix A and column vectors $\mathbf{x}$ and $\mathbf{b}$ should be specified. Solve this system for $\mathbf{x}$. [6]

(b) Solve the differential equation

$$\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}t^2} + M\mathbf{y} = 0,$$

where $\mathbf{y}$ is a vector in $\mathbb{R}^2$, and the matrix M is given by

$$M = \begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}.$$

[6]

(c) (i) Give the necessary and sufficient conditions for a matrix to be a rotation matrix. Use these to show that

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{1}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

is a rotation matrix.

(ii) Consider a triangle with vertices $\mathbf{a} = 4\mathbf{i}$, $\mathbf{b} = 4\mathbf{j}$, $\mathbf{c} = 4\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are orthonormal Cartesian basis vectors. Find the area of the triangle.

(iii) A rotated triangle has vertices $\mathbf{a}' = R\mathbf{a}$, $\mathbf{b}' = R\mathbf{b}$, and $\mathbf{c}' = R\mathbf{c}$. Calculate $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, and determine the area of the rotated triangle. [8]

9. (a) Consider the polynomial

$$f(z) = z^4 - 2z^3 + 3z^2 - 2z + 2,$$

where z is a complex variable.

(i) Find the purely imaginary roots of the equation $f(z) = 0$.

(ii) Hence, or otherwise, find all roots of the equation $f(z) = 0$.

(iii) Show that the sum and product of these roots take the expected values. [7]

(b) (i) Use de Moivre's theorem to show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

(ii) Hence, show that $\tan(\pi/5) = \sqrt{5 - 2\sqrt{5}}$. [6]

(c) (i) Explain why the equation of a plane in $\mathbb{R}^3$ may be written in the form $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{r}$ is the position vector of points in the plane. Use a sketch to define $\mathbf{r}$ and d.

(ii) Find the equation of the line of intersection of the two planes $2x + 3y + 4z = 6$ and $3x + y + 6z = 2$. Express the answer in the vector form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ with a scalar λ, and determine $\mathbf{a}$ and $\mathbf{b}$. [7]

10. (a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{y}{x} + 5.$$

[3]

(b) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y = x^2,$$

that satisfies $y = 0$ when $x = 1$. [5]

(c) Find the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = xe^x,$$

that satisfies $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 0$ when $x = 0$. [8]

(d) Find a solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y\frac{\mathrm{d}y}{\mathrm{d}x},$$

that satisfies $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 1/2$ when $x = 0$. [4]

11. A scalar product is written as $\langle \mathbf{u}, \mathbf{v} \rangle$, for vectors $\mathbf{u}$ and $\mathbf{v}$ from a real vector space. Any scalar product is required to satisfy the following four conditions:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle, \qquad \text{(C1)}$$
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle, \qquad \text{(C2)}$$
$$\langle \mathbf{u}, \lambda \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle, \qquad \text{(C3)}$$
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0, \qquad \text{(C4)}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and real scalars λ, where in (C4) there is an equality if and only if $\mathbf{u} = \mathbf{0}$.

Consider the vector space $\mathcal{S}$ of real symmetric $n \times n$ matrices, where n is a positive integer.

(a) (i) What is the dimension of $\mathcal{S}$?

(ii) Provide a basis for $\mathcal{S}$ for the case where $n = 2$. [3]

(b) For real symmetric $n \times n$ matrices A and B, show that the association

$$\langle A, B \rangle = \operatorname{tr}(AB)$$

satisfies the stated conditions for a scalar product. [6]

(c) If B is also invertible, show that $\operatorname{tr}\left(B^{-1}AB\right) = \operatorname{tr}(A)$. [2]

(d) Prove that there are no real symmetric $n \times n$ matrices A and B such that

$$AB - BA = I,$$

where I is the $n \times n$ identity matrix. [4]

(e) Consider an alternative proposal for a scalar product in $\mathcal{S}$ given by

$$\langle A, B \rangle = \det(AB).$$

Discuss the extent to which this alternative proposal satisfies the conditions (C1) to (C4). [5]