A10285H1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 11 June 2025, 09.30 – 12.00 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find the real and imaginary parts of the following expressions:

(a) $\mathrm{e}^{i\pi/3} - \mathrm{e}^{-i\pi/3}$,

(b) i^i,

(c) $\ln(1+i)$. [6]

2. (a) Show that

$$(|\mathbf{a}| - |\mathbf{b}|)^2 \leq |\mathbf{a}+\mathbf{b}|^2 \leq (|\mathbf{a}| + |\mathbf{b}|)^2,$$

for all vectors $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^3$.

(b) Determine when each of the inequalities in part (a) become an equality. [6]

3. Find all solutions (x, y) to each of the following linear systems:

(a) $2x + 3y = 1, \quad x + 2y = 1.$

(b) $6x + 2y = 1, \quad 12x + 4y = 2.$

(c) $6x + 2y = 2, \quad 12x + 4y = 2.$ [5]

4. (a) For complex numbers z_1 and z_2, show that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

(b) For complex z and real positive c, sketch the curve

$$\mathrm{Re}\left(\frac{z^2}{\mathrm{e}^{i\alpha}}\right) = c^2,$$

in an Argand diagram, for each of the two cases $\alpha = 0$ and $\alpha = \pi/3$. [7]

5. (a) Show that the set of 2×2 Hermitian matrices satisfies all of the properties of a vector space over the field of real scalars.

(b) Demonstrate whether the set of unitary matrices forms a subspace of the vector space described in (a). [7]

6. (a) Find the general solution $y(x)$ of the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}\ln x + \frac{y}{x} = \mathrm{e}^{2x}, \qquad \text{for} \quad x \geq 1.$$

Demonstrate whether this curve can pass through the point with $y = 0$ at $x = 1$.

(b) Find the general solution $y(x)$ of the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(y + x^2\right)^2 - 2x\left(y + x^2\right) + 1}{y + x^2}.$$

You may leave the solution in implicit form $f(x, y) = 0$, for some function f to be determined. [9]

Section B

7. (a) State the properties of eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of a real symmetric $N \times N$ matrix where N is an integer. [3]

(b) (i) Find the characteristic polynomial $p(\lambda)$ for the matrix

$$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

(ii) Show that $p(3) = 0$. Hence determine the eigenvalues, and a set of eigenvectors, of A.

(iii) Determine whether the matrix A is orthogonal, justifying your answer. [10]

(c) (i) By expressing the vector $\mathbf{b} = (1, 1, 1)^T$ in terms of the eigenvectors of A, calculate the value of $A^n\mathbf{b}$ for all values of the positive integer $n \geq 1$.

(ii) Describe the transformation in Cartesian geometry that A represents. [7]

8. (a) A geometric shape known as a parallelepiped lies within the positive quadrant $x > 0$, $y > 0$, and $z > 0$. The parallelepiped is formed from a square lower surface $0 \leq x \leq 2$ and $0 \leq y \leq 2$ in the plane $z = 0$, and a square upper surface in a plane $z =$ constant. The square surfaces are joined by 4 edges, each with vector displacement $\mathbf{d} = (1, 1, 1)^T/\sqrt{3}$. Determine the volume of this parallelepiped. [4]

(b) (i) Next consider a plane P in 3-dimensional Cartesian space, which contains a point with vector position $\mathbf{a}$, and is parallel to vectors $\mathbf{b}$ and $\mathbf{c}$. Determine an equation for the plane in the form

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d,$$

giving expressions for the unit vector $\hat{\mathbf{n}}$ and scalar d in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(ii) Hence find the Cartesian equation of the plane passing through the points $A = (1, 1, 3)$, $B = (0, 2, 1)$ and $C = (0, 1, 2)$. Express the answer in the form

$$\alpha x + \beta y + \gamma z = 1,$$

where the values of α, β and γ should be specified. [8]

(c) Consider a sphere of radius R centred at the origin.

(i) Determine the largest radius $R = R_1$ that is possible without the sphere crossing the plane P considered in part (b)(ii).

(ii) If the sphere radius satisfies $R > R_1$, determine the length of the curve formed by the intersection of the surface of the sphere and the plane P. [8]

9. Consider the ordinary differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = f(t),$$

for the real-valued displacement $x(t)$ as a function of time t, real function $f(t)$, and real positive constants γ and ω with $\gamma < \omega$.

(a) (i) Derive the general solution if $f(t) = 0$.

(ii) Sketch the solution for the case where $x = a$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$. [8]

(b) (i) Derive the general solution to the differential equation if $f(t) = A\cos(\omega t + \phi_0)$ for constants A and ϕ_0.

(ii) Find the root mean square displacement after a time $t \gg 1/\gamma$, and sketch this as a function of ω. [6]

(c) Determine the general solution to the differential equation if $f(t) = B\sin^2\omega t$, where B is constant. [4]

(d) Discuss qualitatively how the solution in each of case (b)(i) and case (c) will change if $\gamma = 0$. [2]

10. (a) Determine the solutions $y(x)$ and $z(x)$ of the system of coupled differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} = 3y + 3z, \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = y + 2z,$$

with initial conditions $y = 1$ and $z = 1$ at $x = 0$. [9]

(b) Consider the second-order differential equation

$$\left(1 - x^2\right) \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + n^2y = 0, \qquad \text{for} \quad -1 \leq x \leq 1,$$

where n is a positive integer.

(i) Using the substitution $x = \cos\theta$ for $0 \leq \theta \leq \pi$, transform this differential equation into the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + b\frac{\mathrm{d}y}{\mathrm{d}\theta} + cy = 0,$$

determining the functions $a(\theta)$, $b(\theta)$ and $c(\theta)$.

(ii) Hence find the general solution $y(x)$ to the differential equation at the start of part (b).

(iii) Sketch the two linearly independent solutions for $n = 1$ and $-1 \leq x \leq 1$. [11]