A14119W1

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SHORT OPTIONS

TRINITY TERM 2024

Tuesday, 11 June

2:30 pm to 4:00 pm

Answer **two** *questions from the option for which you have entered.*

Start the answer to each question in a fresh book.

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

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Section S01 FUNCTIONS OF A COMPLEX VARIABLE

1. a) State Cauchy's Residue Theorem. [3]

b) Find, in the annulus $3 < |z| < \infty$, the Laurent series of the function

$$f(z) = \frac{1}{z^4 + 9z^2}$$

expanded around $z = 0$. [5]

c) Prove that, for the function

$$f(z) = \left(\sin\frac{1}{z}\right)^{-1},$$

$z = 0$ is not an isolated singularity by showing that it has an infinite number of simple poles, accumulating at $z = 0$. [5]

d) Show that, in the limit where the integer $N \to \infty$,

$$\int_{\Gamma_N} \frac{\pi\cot(\pi z)}{(z+a)^2}\,\mathrm{d}z = 0,$$

where the contour Γ_N is a square of side $2N + 1$, whose sides are aligned with the real and imaginary axes and centred on $z = 0$. [6]

e) Use the integral from d) to show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(a+r)^2} = \pi^2\mathrm{cosec}^2(\pi a).$$

[6]

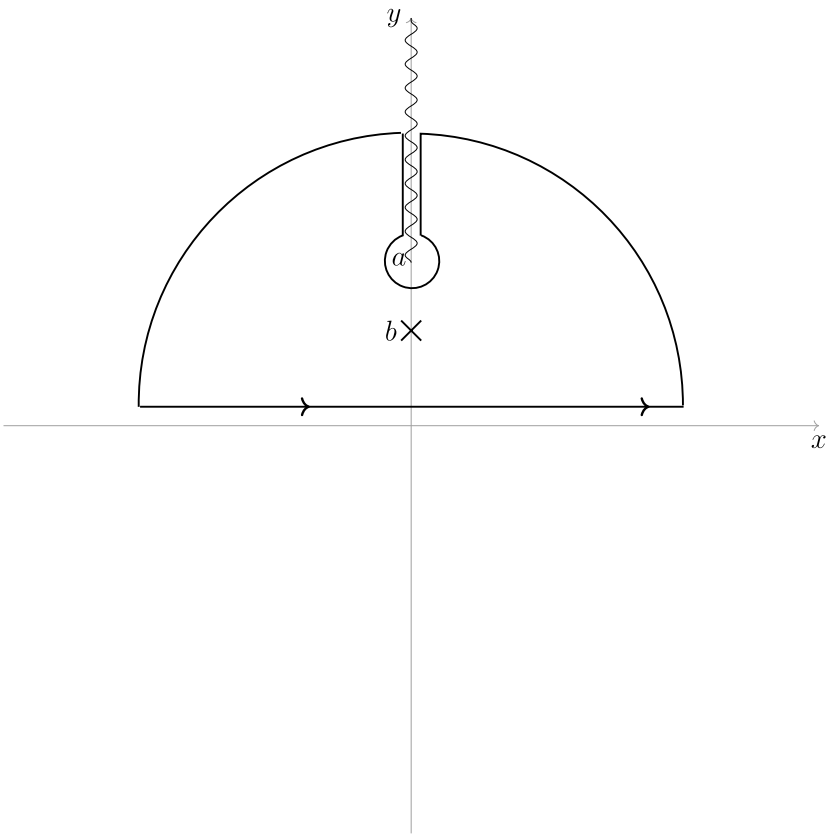
2. a) State Cauchy's Theorem. [3]

b) Describe the singularities of

$$f(z) = \frac{\log\left(a^2 + z^2\right)}{z^2 + b^2}.$$

Define two different sets of branch cuts for this function. [5]

c) Use contour integration, and the contour shown in the figure below,



to show that

$$\int_{-\infty}^{\infty} \frac{\log\left(a^2 + x^2\right)}{x^2 + b^2}\,\mathrm{d}x = \frac{2\pi}{b}\log(a+b),$$

where a and b are positive reals and $a > b$. [7]

d) What contour would you use to calculate the integral for $b > a$? Calculate the contour integral for $b > a$ and show that you obtain the same result as for part c). [5]

e) Use the method of contour integration to find

$$\int_0^{2\pi} \frac{\cos^2 x}{13 + 12\cos x}\,\mathrm{d}x.$$

[5]

3. a) An analytic function, $f(z)$ can be written in terms of its real and imaginary parts as

$$f(z) = u(x, y) + \mathrm{i}v(x, y).$$

Show that u and v must satisfy the Cauchy–Riemann conditions and show that they are harmonic functions. [3]

b) What is a conformal transformation? Prove that an analytic function is a conformal transformation. Show that

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a conformal transformation. Consider the line L given by:

$$\begin{aligned} (x, 0) \qquad & \text{for } 1 < |x| < \infty \\ (\cos\theta, \sin\theta) \qquad & \text{for } 0 \leq \theta \leq \pi\ . \end{aligned}$$

Sketch how $f(z)$ maps the region on or above the line L onto the complex plane. [6]

c) Consider a plate held at a fixed potential, Φ_0 along the line L. Use the conformal transformation from part b) to solve Laplace's equation for Φ in the region above that line. [6]

d) Prove that the field lines, $|\vec{E}| = \text{constant}$, where $\vec{E} = -\vec{\nabla}\Phi$, are perpendicular to the lines of constant Φ and sketch them for this configuration. [5]

e) Use the fact that $\sin^2(z)$ is an analytic function to calculate

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\pi}{6} + 2\mathrm{e}^{\mathrm{i}\theta}\right)\,\mathrm{d}\theta.$$

[5]

Section S02 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. a) Name and briefly describe five of the evolutionary stages of a star with an initial mass of three solar masses after it leaves the main sequence. Draw a simple Hertzsprung-Russell diagram and identify those stages on it. [11]

b) Assume that the Sun is a perfectly spherical object, and that the radial profile of the mass volume density ρ within it is given by $\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^n$, where r_0 is a scale radius, n the power-law index, and ρ_0 a normalisation constant. Also assume we know the solar radius $\mathrm{R}_\odot$, total solar mass $\mathrm{M}_\odot$, and the pressure at the centre of the Sun $\mathrm{P}_{\mathrm{c},\odot}$. Use the equations of stellar structure to derive an expression for ρ_0 and the radial profile of the pressure within the Sun $P_\odot(r)$. [6]

c) What are the constraints required on n to obtain a reasonable model of the solar interior? Assuming $n = \frac{1}{2}$ and $r_0 = \frac{\mathrm{R}_\odot}{2}$, derive an estimate of the central pressure of the Sun. How does this compare to simple order-of-magnitude estimates (e.g. as derived by assuming average or typical quantities for each term of the hydrostatic equilibrium equation)? [8]

[Assume the total mass of the Sun is $\mathrm{M}_\odot = 2.0 \times 10^{30}\,\mathrm{kg}$ and the radius of the Sun is $\mathrm{R}_\odot = 7.0 \times 10^{8}\,\mathrm{m}$.]

2. a) For each of elliptical and spiral galaxies, state what the classification of the galaxies is along the Hubble Sequence, and state which property(ies) determine the exact galaxy type. [10]

b) A galaxy with a total apparent V-band magnitude of 8.5 and effective (i.e. half-light) radius of 30 arcseconds is composed of a dominant spheroid of stars with random motions and a tenuous gaseous disc in circular rotation. The rotation velocity of the gas at large radii and the velocity dispersion of the stars averaged within the V-band effective radius are both $315\,\mathrm{km\,s^{-1}}$.

Assume the galaxy obeys the Tully–Fisher relation of disc galaxies,

$$\mathrm{M}_V = -8.0 \left[\log_{10} \left(\frac{V_\mathrm{rot}}{\mathrm{km\,s^{-1}}} \right) - 2.5 \right] - 22.5 \ ,$$

where M_V is the total absolute magnitude of a galaxy in the V band and V_rot is the asymptotic galaxy rotation velocity. Determine whether the galaxy also obeys the fundamental plane of spheroids,

$$\log_{10} \left(\frac{R_{V,\mathrm{e}}}{\mathrm{kpc}} \right) = 1.2 \log_{10} \left(\frac{\sigma_\mathrm{e}}{\mathrm{km\,s^{-1}}} \right) - 0.8 \log_{10} \left(\frac{\Sigma_{V,\mathrm{e}}}{\mathrm{L}_{V,\odot}\ \mathrm{kpc^{-2}}} \right) + 4.95 \ ,$$

where $R_{V,\mathrm{e}}$ is the V-band effective radius, σ_e is the average stellar velocity dispersion within 1 $R_{V,\mathrm{e}}$, $\Sigma_{V,\mathrm{e}}$ is the average V-band luminosity surface density within 1 $R_{V,\mathrm{e}}$, and $\mathrm{L}_{V,\odot}$ is the V-band luminosity of the Sun. [11]

c) A supernova explodes in the same galaxy, leading to a doubling of the galaxy V-band luminosity for a period of about one month. Estimate a size from this, and discuss your result in view of the galaxy size that can be inferred in part b). [4]

[Assume the absolute V-band magnitude of the Sun is $\mathrm{M}_{V,\odot} = 5.0$.]

3. a) State the main pieces of observational evidence for the Big Bang. [3]

b) What properties should any dark matter candidate possess? What is the most likely candidate for dark matter? [5]

c) Write the Friedmann equation for an expanding universe with only matter (i.e. no radiation nor cosmological constant) and a negative curvature $k = -5.0 \times 10^{-8}\,\mathrm{Mpc^{-2}}$. Assuming the curvature term dominates, work out the dependence of the scale factor and of the Hubble constant on time. What is the age of that universe today? What is the expansion rate of that universe today, and what was it when that universe was only 10% of its current age? [7]

d) What is the matter density of that same universe today? The farthest galaxies are observed to be at a redshift of 9. What was the matter density at that redshift? [6]

e) In that same universe, is it possible for the energy density associated with matter and curvature to be equal, and if so when? [4]

[Assume that the Hubble constant today $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.]

Section S03 QUANTUM IDEAS

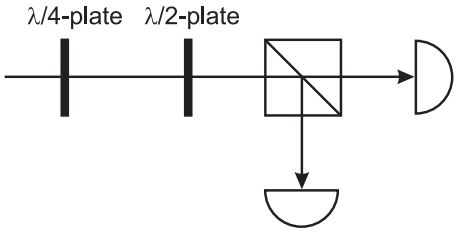
1. (a) Write a general formulation of the measurement postulate of quantum mechanics for a single Hilbert space. [4]

(b) Consider two polarisation states of the photon:

$$|\psi\rangle = \frac{3|H\rangle + \mathrm{i}|V\rangle}{\sqrt{10}}, \quad |\phi\rangle = \frac{\mathrm{i}|H\rangle + 3|V\rangle}{\sqrt{10}},$$

where $|H\rangle$ and $|V\rangle$ denote the orthonormal horizontal and vertical polarisation states. Show that these states form an orthonormal basis. [3]

(c) An apparatus for measuring monochromatic photon polarisation states in the basis $\{|\psi\rangle, |\phi\rangle\}$ consists of two waveplates, a polarising beam splitter and two photon detectors as shown in the diagram (top view). Determine the angles of the two waveplates' optic axes with respect to the horizontal that would enable this measurement. Describe how each of the basis states is transformed by the waveplates. [5]



(d) Express the state $|\chi\rangle = (|H\rangle - 2\mathrm{i}|V\rangle)/\sqrt{5}$ in the basis $\{|\psi\rangle, |\phi\rangle\}$. Hence find the probabilities of the possible outcomes if this state is measured in that basis. [3]

(e) We define an observable operator $\hat{\sigma}$ for the measurement in the $\{|\psi\rangle, |\phi\rangle\}$ basis by associating state $|\psi\rangle$ with value 1, and $|\phi\rangle$ with value -1. Write the matrix of $\hat{\sigma}$ in the canonical basis $\{|H\rangle, |V\rangle\}$. [2]

Find the expectation value and root mean square uncertainty of measuring $\hat{\sigma}$ in the state $|\chi\rangle$. [4]

(f) The states $\{|\psi\rangle, |\phi\rangle\}$ comprise one of the communication bases in the BB84 quantum protocol. What requirements does a pair of states $\{|\psi'\rangle, |\phi'\rangle\}$ need to satisfy in order to serve as the second basis in that protocol? Propose a set of states that could comprise the second basis. [4]

2. (a) Define the notion of quantum entanglement. [2]

(b) Alice and Bob share two qubits in the state

$$|\Psi\rangle = \mathcal{N}(|00\rangle + 2\mathrm{i}|10\rangle - 3\mathrm{i}|11\rangle).$$

What is the normalisation factor $\mathcal{N}$? [1]

Is state $|\Psi\rangle$ separable or entangled? Prove your answer. [6]

(c) Alice performs a local measurement of state $|\Psi\rangle$
(i) in the canonical basis $\{|0\rangle, |1\rangle\}$;
(ii) in the basis

$$\left\{ |R\rangle = \frac{|0\rangle + \mathrm{i}|1\rangle}{\sqrt{2}},\ |L\rangle = \frac{|0\rangle - \mathrm{i}|1\rangle}{\sqrt{2}} \right\}.$$

For each of these bases, find the probability of each possible outcome of Alice's measurement. For each case, write down the (normalized) state in which Bob's qubit will be remotely prepared after Alice's measurement. [8]

It would appear that Alice's choice of basis affects the state remotely prepared with Bob, and can hence be used to instantly communicate information between the two parties. Is this conclusion correct? Explain your answer. [4]

(d) Describe the von Neumann model of quantum measurements. Explain how it leads to the many-worlds interpretation of quantum mechanics. [4]

3. (a) The evolution operator $\hat{U}(t)$ acts on an arbitrary eigenstate $|E_i\rangle$ of the Hamiltonian (corresponding to the energy E_i) as follows:

$$\hat{U}(t)|E_i\rangle = \exp\left(-\frac{\mathrm{i}}{\hbar}E_i t\right)|E_i\rangle.$$

(i) Write down the evolution operator in the energy basis $\{|E_1\rangle, \ldots, |E_N\rangle\}$ in both the Dirac and matrix forms. [3]

(ii) Express this operator in terms of the Hamiltonian $\hat{H}$. Use this to derive the (time-dependent) Schrödinger equation. [5]

(b) An atom has two energy levels $|a\rangle$ and $|b\rangle$ with energies 0 and E (where $E > 0$), respectively.

(i) Write down the matrix of the corresponding Hamiltonian $\hat{H}_0$ in the basis $\{|a\rangle, |b\rangle\}$. [1]

At time $t = 0$, a perturbation $\hat{V} = \frac{2}{3}E\left(|a\rangle\langle b| + |b\rangle\langle a|\right)$ is applied, so the Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{V}$.

(ii) Calculate the corresponding evolution operator. [7]

(iii) Write down the state $|\psi(t)\rangle$ of the atom at an arbitrary time t if it is initially in the ground state $|a\rangle$. [3]

(iv) Find the probability to find the atom in the excited state $|b\rangle$ as a function of time. Your answer must be simplified to eliminate the imaginary unit i. [6]