A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Monday 2nd September 2024, 9.30 am – 12 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Evaluate the following limits:

(a)
$$\lim_{x\to 1} \frac{3^x - 3}{x - 1},$$
[2]

(b)
$$\lim_{x\to 0} \left(\ln\left(1 + x\right)\right)^x .$$
[3]

2. The properties of an infinite string under tension are altered so that the wave equation describing small amplitude transverse waves on the string becomes

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = -\Omega^2 y(x,t),$$

where y is the transverse displacement of the string, x is the position along the string, t is the time, and c and Ω are real constants. By utilising the *ansatz* $y(x,t) = \mathrm{Re}[\exp i(\omega t \pm kx)]$ or otherwise, find the relation that the modified wave equation implies between the wave number k and angular frequency ω. Compute the phase velocity v_p and group velocity v_g of the waves as a function of wave number, and comment on the relation of these to c. What are the limiting behaviours of both v_p and v_g as $k \to 0$ and $k \to \infty$? [6]

3. Use the divergence theorem to evaluate

$$I = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

where

$$\mathbf{F} = \sin(\pi x)\hat{\mathbf{i}} + zy^3\hat{\mathbf{j}} + \left(z^2 + 4xy\right)\hat{\mathbf{k}},$$

and S is the surface of the box bounding the volume $-1 \le x \le 1$, $0 \le y \le 1$ and $1 \le z \le 4$. All six sides of the box are included in S. [6]

4. Consider a function f in the (x, y) plane given by $f(x, y) = y^3 + 2xy$. Find the unit vector $\hat{\mathbf{b}}$ along which the derivative of f, evaluated at $(x, y) = (1, 1)$, is of greatest magnitude. In which directions is f stationary at this point? [5]

5. Let $f_{X,Y}(x, y) = A \sin x \cos y$ define a joint probability density function normalised over the region bounded by the lines $y = \pi/2 - x$, $y = \pi/2$ and $x = \pi/2$. Find the value of the constant A and hence the expected value of the function $g(y) = 1/\cos y$ over this region. [7]

6. Calculate the length of the curve $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ for $0 < t < \pi$. [4]

7. Suppose $xy^3 + yz^2 = 1$ and $v = x^2y^3$, where x, y, z and v are all real. Using a chain rule, express $(\partial v/\partial z)_x$ as a function of x, y and z, and evaluate it for $(x, y, z) = (1, 1, 1)$. [7]

Section B

8. (a) Show that $\nabla \times \mathbf{G} = \mathbf{0}$ for the vector field

$$\mathbf{G} = 2xyz\nabla x + x^2 z \nabla y + x^2 y \nabla z,$$

where (x, y, z) are Cartesian coordinates. [2]

(b) Given the result from part (a), explain why $\mathbf{G} \cdot \mathrm{d}\mathbf{r}$ is an exact differential of a scalar function $\phi(x, y, z)$, where $\mathrm{d}\mathbf{r}$ is a vector line element, and obtain an explicit expression for ϕ. [4]

(c) Evaluate $\int \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ by integrating along the following two paths: i) along the curve $(x, y, z) = (t^3, t^2, 2t)$, where t varies from $t = 0$ to $t = 1$; and ii) along a path that proceeds first along the y-axis, then the z-axis, and finally the x-axis, from the origin to the point $(1, 1, 2)$. [4]

(d) Compare the results from part (c) with the difference in the value of ϕ at the two ends of each path and comment on your results. [4]

(e) State Stokes' theorem for the case of a vector field $\mathbf{F}$ over a surface S with a closed boundary C. Verify Stokes' theorem for the vector field

$$\mathbf{F} = 2xz^2\nabla x + (3y - z)\nabla y + 2x^2 z \nabla z,$$

where S is the surface parameterised by $x^2 + y^2 + z^2 = 4$ and $x > 0$, and C is its bounding curve in the yz plane. [6]

9. (a) Sketch the cylindrical and spherical polar coordinate systems and give expressions for the Cartesian components x, y, z in each system. [4]

(b) Consider a thin, hemispheric shell corresponding to the $z > 0$ half of the sphere $x^2 + y^2 + z^2 = 1$. If the surface density is given by $\sigma(x, y, z) = z^2$, what is the mass of the shell? [6]

(c) Evaluate

$$I = \iint_S (x + z)\, \mathrm{d}S,$$

where S is the surface whose side is the cylinder $x^2 + y^2 = 1$, whose bottom is the disc $x^2 + y^2 \leq 1$ in the xy-plane and whose top is the plane $z = 2 - x$. [10]

10. (a) Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

that describes the small-amplitude transverse displacement y of a uniform string of mass per unit length ρ and tension T. Here x and t denote position and time, respectively, and you may assume that gravity is negligible. Indicate from your derivation how the wave phase speed c is related to the properties of the string. [4]

(b) Show that the energy transported per unit time by a sinusoidal wave on a uniform string is

$$P = \frac{1}{2} T \omega k A^2,$$

where ω is the angular frequency of the wave, k is its wave number and A is its amplitude. [5]

(c) A string of mass per unit length ρ_1 is connected to a second string of mass per unit length ρ_2, where $\rho_2 = 9\rho_1$. A sinusoidal wave propagates along the lighter string towards the boundary between the strings, where some of the wave energy is transmitted into the more massive string, and some is reflected.

Obtain expressions for the amplitudes of the transmitted and reflected waves, and hence find the values of the transmission and reflection coefficients. Explain your choice of boundary conditions. [4]

(d) Use the results from part (b) to find: (i) the ratio of the transmitted to incident power; and (ii) the ratio of the reflected to incident power, in the string system from part (c). Show that the sum of the reflected and transmitted power equals the incident power. [3]

(e) Describe what would happen if (i) $\rho_1 = \rho_2$, (ii) $\rho_1 \ll \rho_2$, and (iii) $\rho_1 \gg \rho_2$ in the string system of part (c). Give an interpretation of the sign of the reflection and transmission coefficients in each of these cases. [4]

11. Two particles of mass m and $2m$ are free to slide on a frictionless horizontal circular wire of radius r. The particles are connected by two identical massless springs of spring constant k and natural length πr, which also wind around the wire, as shown in the figure.



(a) Write down the equations of motion for the displacements along the wire, $x_1(t)$ and $x_2(t)$, of the two particles. [2]

(b) Find the frequencies of the normal modes of the system, and qualitatively explain their values. [6]

(c) Find the functional forms of the motions associated with each of the normal modes, including the appropriate time dependence. [7]

(d) At time $t = 0$, the displacements and velocities of the two particles are given by $x_1(0) = \pi r/4$, $x_2(0) = -\pi r/8$, and $\dot{x}_1(0) = \dot{x}_2(0) = \pi\omega r$, where ω is the largest of the frequencies found in (b). Find the first time when the velocity of the particle of mass $2m$ is zero. Write your answer in terms of ω. [5]