A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 6 June 2024, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) Evaluate L_1, where

$$L_1 = \lim_{x \to \pi} \frac{\cos 2x - 1}{(x - \pi)^2}.$$ [2]

(b) Evaluate L_2, where

$$L_2 = \lim_{x \to 0} (\sin x)^{1/\ln x}.$$ [3]

2. Find all stationary points of the function $f(x, y) = x^2 + 2x - 2y^3 + 6y^2$, where $(x, y) \in \mathbb{R}^2$. Classify the stationary points as maxima, minima or saddle points. [4]

3. Calculate the area A_Ω of the surface Ω parameterised by

$$b\left(x^2 + y^2\right) - z = 0 \quad \text{and} \quad z \leq 4b,$$

where (x, y, z) are Cartesian coordinates and $b > 0$ is a real constant. [6]

4. The angular frequency ω and wave number k of certain waves in a magnetised plasma (called cyclotron waves) obey the following relation, known as a dispersion relation:

$$\omega^2 = \Omega^2 + k^2 c_s^2,$$

where c_s is the sound speed and Ω is the cyclotron frequency. Neither c_s nor Ω depend on k.

(a) Derive expressions for the phase velocity v_p and group velocity v_g of the cyclotron waves. [2]

(b) Sketch v_p and v_g versus k. What is the characteristic scale in k that sets the boundary between the 'long-wavelength' and 'short-wavelength' limits? [4]

(c) Two wave packets – one composed of long-wavelength plane waves and another composed of short-wavelength plane waves – are launched into a magnetised plasma. Discuss qualitatively how the shapes of the two wave packets will change with time as they travel through the plasma. [2]

5. Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F} = (-yz, x^2 z, xy^2)$, where the curve C starts at $(0, 0, 0)$, ends at $(1, 1, 1)$ and is defined by the intersection of the plane $x = y$ with the surface $z = x^2$. Here (x, y, z) are Cartesian coordinates. [5]

6. A joint probability density function, or PDF, for the continuous random variables X and Y is given by

$$f_{XY}(x, y) = Ax^2 \exp\left(-x^2 - y^2\right),$$

where x and y are real numbers in the interval $(-\infty, \infty)$.

(a) Determine the constant A. [2]

(b) By transforming to plane polar coordinates, or otherwise, calculate the probability of a particular event satisfying $0 \leq X < \infty$ and $|Y| \leq X$. [4]

You may find the following integrals useful:

$$\int_{-\infty}^{\infty} s^{2n} \exp\left(-s^2\right) \mathrm{d}s = \frac{(2n)!}{4^n n!}\sqrt{\pi}, \quad \int_{0}^{\infty} s^{2n+1} \exp\left(-s^2\right) \mathrm{d}s = \frac{n!}{2},$$

where $n \geq 0$ is an integer.

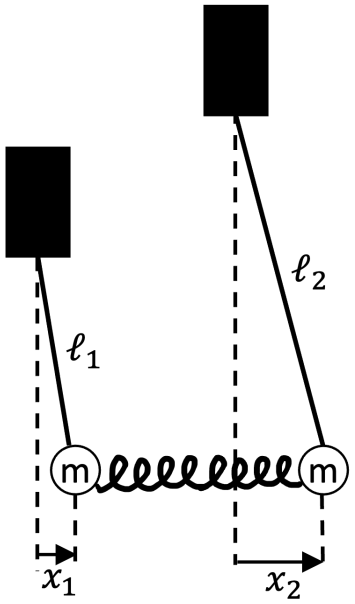
7. Calculate the Jacobian determinant of the transformation $(x, y) = (v, v/u)$ and use it to evaluate the integral

$$I = \iint \left(\frac{x^2}{y^2}\right) e^{x/y} \, \mathrm{d}x \, \mathrm{d}y,$$

where x and y are real numbers, and the region of integration is $0 \leq y \leq 1$ and $0 \leq x \leq y$. [6]

Section B

8. Two simple pendulums with massless wires of lengths ℓ_1 and ℓ_2, respectively, are attached to point particles of mass m, both at the same equilibrium height. The point masses are attached by a massless spring with spring constant k and natural length chosen so that in equilibrium the pendulum wires are vertical. The displacements of the two particles from their equilibrium positions are x_1 and x_2, as shown below.



(a) Assuming x_1 and x_2 are small, write down the equations of motion for the two particles in terms of $\omega_1^2 = g/\ell_1$, $\omega_2^2 = g/\ell_2$ and $\Omega^2 = k/m$, where g is the acceleration due to gravity. Clearly state what it means for x_1 and x_2 to be small. [5]

(b) Calculate the system's normal mode frequencies in terms of Ω^2, $\tilde{\omega}_1^2 = \omega_1^2 + \Omega^2$ and $\tilde{\omega}_2^2 = \omega_2^2 + \Omega^2$. [5]

(c) The pendulum lengths are now specified to be $\ell_2 = 2\ell_1 = gm/k$. For each of the normal modes obtained in (b), calculate the ratio of particle displacements x_2/x_1. [5]

(d) The particles are released from rest at $t = 0$ with initial positions $x_1 = 0$ and $x_2 = \sqrt{5}$. Solve for their positions at all subsequent times $t > 0$. [5]

9. A string of uniform mass per unit length ρ and tension T is held fixed at $x = 0$ and $x = L$. The transverse displacement of the string $y(x, t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0,$$

where $c^2 = T/\rho$.

(a) Derive expressions for the allowed normal mode frequencies and wavenumbers. [6]

(b) At time $t = 0$, the string is released from rest with an initial transverse displacement

$$y_0(x) = \sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right).$$

Calculate the transverse displacement of the string at subsequent times. [5]

(c) Derive a general expression for the potential energy of the string due to its transverse displacement. [5]

(d) Given the initial conditions from (b), what is the ratio of energies stored in each normal mode at time t? [4]

10. (a) State Stokes' Theorem, precisely defining all terms. [4]

(b) Verify Stokes' Theorem for the vector field

$$\mathbf{F}(x, y, z) = (y^2 - y, 2xy, z)$$

defined on the surface S_1 parameterised by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad \text{and} \quad z = 0,$$

where $a > 0$ and $b > 0$ are real constants. [10]

(c) Evaluate

$$\int_{S_2} (\nabla \times \mathbf{G}) \cdot \mathrm{d}\mathbf{S}_2$$

for the vector field

$$\mathbf{G}(x, y, z) = \mathbf{F}(x, y, z) + (xz, 2yz, 0)$$

defined on the surface S_2 parameterised by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad z \geq 0,$$

where a, b and $\mathbf{F}$ are as given in part (b), and $c > 0$ is a real constant. Justify your result. [6]

11. Consider the function $\rho = \rho(t, x(t), y(t), z(t))$, where (x, y, z) are Cartesian coordinates and t is time.

(a) Write $\mathrm{d}\rho/\mathrm{d}t$ in terms of partial derivatives with respect to x, y, z and t. Be sure to state clearly which variables are held fixed for all partial derivatives. [4]

(b) Show that

$$\nabla \cdot (\phi \mathbf{A}) = \mathbf{A} \cdot \nabla\phi + \phi \nabla \cdot \mathbf{A},$$

where ϕ is any scalar and $\mathbf{A}$ is any vector. [3]

(c) Use your result from (a) to express $\mathrm{d}\rho/\mathrm{d}t$ in terms of ρ and $\mathbf{u} = (\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, \mathrm{d}z/\mathrm{d}t)$. Show that when $\mathrm{d}\rho/\mathrm{d}t = 0$ and $\nabla \cdot \mathbf{u} = 0$, the time derivative of ρ at fixed (x, y, z) can be written as the divergence of a vector. [4]

(d) At $t = t_0$, we have

$$\mathbf{u}(t_0, \mathbf{r}) = u_0 \frac{a^2}{r^2} \nabla r, \quad \rho(t_0, \mathbf{r}) = \rho_0 e^{-(r-r_0)^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the magnitude of the position vector $\mathbf{r}$, and u_0, ρ_0 and r_0 are real constants, with $\rho_0 > 0$ and $r_0 > 0$. Using the divergence theorem (or otherwise), calculate the time derivative of

$$N_V = \int_V \mathrm{d}^3\mathbf{r}\ \rho(t, \mathbf{r})$$

at $t = t_0$. The integration volume V is a sphere of radius a, centred at the origin. [7]

(e) Discuss the physical significance of the result in (d) for the case where ρ is the mass density of a fluid. [2]