A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Tuesday 3rd September 2024, 9:30 am – 12 noon

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Prove the identity

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)}{\sin(x/2)} \cos((N-1)\frac{x}{2}).$$

[6]

2. Show that, for a real number $a > 0$,

$$\frac{ia-1}{ia+1} = e^{2i\theta},$$

where $\tan\theta = \dfrac{1}{a}$, and $0 < \theta < \dfrac{\pi}{2}$. [6]

3. Solve the following differential equations

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{(x+1)} = (x+1)^3,$$

[6]

(ii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \mathrm{e}^x \sin x.$$

[6]

4. Three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form the sides of a triangle. Show that

$$\mathbf{a} \times \mathbf{b} = \pm\mathbf{b} \times \mathbf{c} = \pm\mathbf{c} \times \mathbf{a}.$$

[4]

5. The equations $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ and $(\mathbf{r} - \mathbf{c}) \cdot \mathbf{d} = 0$ represent a line and a plane, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are constant vectors. Show that if $\mathbf{b} \cdot \mathbf{d} \neq 0$, the line will intersect the plane exactly once, and find the point of intersection. [4]

6. (i) An upper triangular matrix with elements a_{ij} satisfies $a_{ij} = 0$ for all $i > j$. If $\mathbf{A}$ and $\mathbf{B}$ are both upper triangular $n \times n$ matrices, prove that $\mathbf{AB}$ is upper triangular and find its diagonal elements.

(ii) Find an upper triangular matrix $\mathbf{A}$ such that

$$\mathbf{A}^3 = \begin{pmatrix} 27 & -74 \\ 0 & 64 \end{pmatrix}.$$

[8]

Section B

7. The displacement of a linear forced damped oscillator is given by the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = A\sin(\omega t),$$

where the constants a and b are greater than zero.

(a) Obtain the solution when $A = 0$, showing how it takes different forms depending on the relative values of a and b. [8]

(b) For $A \neq 0$, obtain the solution which oscillates with a frequency ω, and determine the amplitude as a function of ω. [6]

(c) Provided $b > \frac{1}{2}a^2$, this amplitude attains its greatest value for fixed forcing amplitude A when

$$\omega^2 = \alpha b - \beta a^2.$$

Determine the constants α and β. [6]

8. The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Compute the inverse of $\mathbf{B}$. You may use the fact that $\mathbf{B}^{-1} = \mathrm{adj}(\mathbf{B})/\det(\mathbf{B})$, where $\mathrm{adj}(\mathbf{B})$ is the transpose of the co-factor matrix of $\mathbf{B}$. [4]

(b) The matrix $\mathbf{D}$ is defined by $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$. Show that $\mathbf{D}$ is a diagonal matrix of the form

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

[6]

(c) Derive an expression for $\mathbf{D}^n$, where n is a positive integer, in terms of d_1, d_2, d_3 and find an expression for $\mathbf{A}^n$ in terms of $\mathbf{D}^n, \mathbf{B}$ and $\mathbf{B}^{-1}$. [4]

(d) Use the definition of matrix $\mathbf{D}$ to find the inverse of matrix $\mathbf{A}$. [6]

9. A biological process involving three types of protein is characterised by their concentrations $A(t), B(t)$ and $C(t)$ respectively. The concentrations obey the following differential equations

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -\epsilon A + \delta C, \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= -\eta B + \epsilon A, \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= -\delta C + \eta B, \end{aligned}$$

where ϵ, δ and η are positive real numbers.

(i) Find the matrix $\mathbf{M}$ for which the three differential equations can be written in the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{M}\,\mathbf{x},$$

where the vector

$$\mathbf{x}(t) = \begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix}.$$

[2]

(ii) Show that two of the eigenvalues of $\mathbf{M}$ can be written in the form

$$\lambda_{\pm} = K \pm \sqrt{K^2 - W},$$

and find the values of the constants K and W. [5]

(iii) Find the third eigenvalue, λ_3, and the corresponding eigenvector. [4]

(iv) Write down the form of $\mathbf{M}$ in a new basis in which it is diagonal and in which

$$\mathbf{x}'(t) = \begin{pmatrix} \mu_1\, e^{-\rho_1 t} \\ \mu_2\, e^{-\rho_2 t} \\ \mu_3 \end{pmatrix},$$

where μ_1, μ_2 and μ_3 are constants. Find the values of the constants ρ_1 and ρ_2. [4]

(v) Hence or otherwise, find the equilibrium (steady state) concentration of each of the proteins, given the initial conditions $A(0) = n,\ B(0) = 0$, and $C(0) = 0$. [5]

10. (a) The exponential of a matrix $\mathbf{A}$ is defined as

$$\exp(\mathbf{A}) = \sum_{m=0}^{\infty} \frac{\mathbf{A}^m}{m!}.$$

Show that if $\mathbf{A}$ is Hermitian, then the exponential of $\mathbf{A}$ is also Hermitian. Find the matrix elements of the exponential of $\mathbf{A}$ in terms of its matrix elements for the case where $\mathbf{A}$ is a diagonal matrix. [4]

(b) By diagonalizing the matrix, or otherwise, find $\exp(\mathbf{A})$ for the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}.$$

[6]

(c) An idempotent matrix $\mathbf{M}$ is a matrix for which $\mathbf{M}^2 = \mathbf{M}$.
For a general idempotent matrix $\mathbf{M}$, show that:

(i) $\mathbf{M}$ must be a square matrix;

(ii) if $\mathbf{M}$ is invertible, it must be the identity matrix $\mathbf{I}$;

(iii) $\mathbf{I} - \mathbf{M}$ is also idempotent;

(iv) $\det(\mathbf{M}) = 0$ or 1;

(v) if $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal unit vectors, $\mathbf{M} = \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T$ is idempotent;

(vi) $\exp(\alpha\mathbf{M}) = \mathbf{I} + \mathbf{M}(\exp(\alpha) - 1)$, where α is a scalar. [7]

(d) Show that a necessary condition for a 2×2 matrix to be idempotent is that it is diagonal or that its trace equals 1. [3]