A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 5 June 2024, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

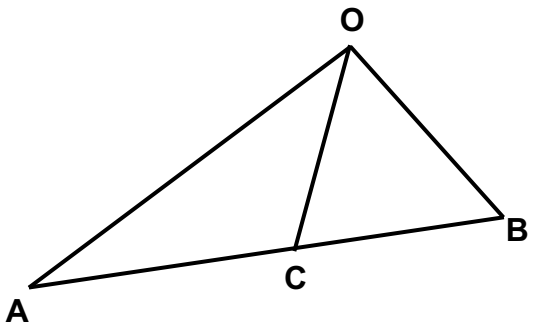
The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Point C lies on side AB of triangle OAB, as shown below:



Represent side OA with the vector $\mathbf{a}$, and side OB with the vector $\mathbf{b}$. Draw a diagram to explain why OC can be represented by a vector $\mathbf{c} = \mathbf{a} + x(\mathbf{b} - \mathbf{a})$, where x is a scalar. Hence derive an expression for x in terms of $\mathbf{a}$ and $\mathbf{b}$ when OC is perpendicular to AB. [3]

2. (i) Sketch, on an Argand diagram, the points $z = x+iy$ that satisfy the conditions (a) $|z+i| = 1$ and (b) $|z+i| = |z-1|$, and find the values of z where the two graphs intersect. [3]

(ii) Find all solutions of the equation $x^4 + 4x^2 + 16 = 0$. [4]

(iii) Sketch the locus of the variable z in the complex plane, where z satisfies

$$\arg\left((z+i)^2\right) = -\frac{3\pi}{2}.$$

[3]

3. Let $\mathbf{A}$ be a 3×3 anti-symmetric matrix. Show that the linear equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x}$ and $\mathbf{b}$ are 3×1 column vectors, does not, in general, have a unique solution. [5]

4. The displacement $x(t)$ of a mass in a damped mass-spring system satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2K\frac{\mathrm{d}x}{\mathrm{d}t} + K^2x = 0,$$

where K is a real positive constant. At time $t = 0$, the mass has zero displacement, but is moving with velocity v_0. Solve the above equation for $x(t)$. [4]

5. A nearby reservoir initially contains one unit of pollutant. A chemical plant deposits pollutant into the reservoir at a rate proportional to half the instantaneous amount of pollutant present in the reservoir, with proportionality constant k. In addition, a pump continually extracts pollutant at a rate of a third of a unit per hour.

Write down a differential equation for the amount of pollutant $p(t)$ contained in the reservoir, and solve it to obtain an expression for p as a function of time. [5]

6. A parallelepiped, a three-dimensional volume whose faces are all parallelograms, is uniquely specified by three vectors originating from a common point. Consider three vectors, all originating from the origin, with endpoints at $(-5, 3, -3)$, $(3, -2, 1)$ and a point in the plane $3x+4y-z = k$, with $k \neq 0$. Calculate the volume of the parallelepiped thus specified. [6]

7. A matrix is in reduced row echelon form if it satisfies the following conditions:

- All rows having only zero entries are at the bottom.
- The leading entry (left-most non-zero entry) of every non-zero row is 1, and is to the right of the leading entry of every row above.
- Each column containing a leading 1 has zeros in all its other entries.

Employ Gaussian elimination to convert the matrix $\mathbf{M}$ augmented by the identity matrix, $\mathbf{M}|\mathbf{I}$ to reduced row echelon form, where

$$\mathbf{M}|\mathbf{I} = \left(\begin{array}{ccc|ccc} 0 & 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right).$$

[6]

What is the inverse of the square matrix $\mathbf{M}$? [1]

Section B

8. Consider the matrix $\mathbf{B}$ defined as

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 1 \\ -3 & 0 & 1 \end{bmatrix}.$$

(a) Determine a lower-triangular matrix $\mathbf{L}$ with unit entries along its diagonal, and an upper-triangular matrix $\mathbf{U}$, such that $\mathbf{LU} = \mathbf{B}$. This process is known as an L−U decomposition of $\mathbf{B}$. [7]

(b) Using the L−U decomposition from part (a), calculate the determinant of $\mathbf{B}$. [3]

(c) Employing the L−U decomposition of $\mathbf{B}$, or otherwise, compute column vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ that solve the equations

$$\mathbf{B}\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{B}\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

[8]

(d) What is the significance of the column vectors computed in part (c)? [2]

9. Let $\mathbf{A}$ be a real anti-symmetric matrix. How is $\mathbf{A}$ related to $\mathbf{A}^{\mathrm{T}}$, and to $\mathbf{A}^{*}$? [2]

(a) Show that the matrix $\mathbf{M} = i\mathbf{A}$ is Hermitian. State the properties of (i) the eigenvalues of $\mathbf{M}$ and (ii) the eigenvectors of $\mathbf{M}$ that correspond to distinct eigenvalues. [4]

(b) Given that $\mathbf{v}$ is an eigenvector of $\mathbf{M}$ corresponding to eigenvalue λ, show that $\mathbf{v}$ is also an eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\mu = -i\lambda$. Using this result and the observations from part (a), state general properties of the eigenvalues and eigenvectors of an anti-symmetric matrix. [4]

(c) Compute the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}.$$

Show that the eigenvectors are mutually orthogonal. [10]

10. Every point $\mathbf{x}$ on a planar surface in three dimensions satisfies the relation $\mathbf{x}\cdot\hat{\mathbf{n}} = d$, where $\hat{\mathbf{n}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is the unit vector normal to the plane, and a, b, c, and d are real constants.

(i) The position vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ describe three points that all lie on the plane, with

$$\begin{aligned}\mathbf{u} &= \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}} \\ \mathbf{v} &= 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} \\ \mathbf{w} &= 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}\end{aligned}$$

Find d, and the components of the unit normal $\hat{\mathbf{n}}$. [5]

(ii) Find the point on the plane that is closest to the origin. [5]

(iii) Give a general expression for the point $\mathbf{p}$ that lies within the plane, and is closest to an arbitrary point $\mathbf{q}$, where $\mathbf{q}$ is not in the plane. (hint: you may wish to consider $\mathbf{q}$ to be the origin). [5]

(iv) Identify a set of three mutually perpendicular unit vectors ($\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$), two of which lie within the plane, that form a right-handed reference frame. Find the coordinates of the point $(3, 3, 3)$ in this new reference frame. [5]

11. (i) Find the solution to the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 3\frac{\mathrm{d}u}{\mathrm{d}t} + 2u = 2t + 4,$$

where at $t = 0, u = 4$ and $\dfrac{\mathrm{d}u}{\mathrm{d}t} = -2$. [6]

(ii) A system described by two parameters $x(t)$ and $y(t)$ is governed by a pair of coupled differential equations which can be written in matrix form as

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\ddot{\mathbf{w}} + \begin{pmatrix} 3 & 0 \\ -3 & 0 \end{pmatrix}\dot{\mathbf{w}} + \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}\mathbf{w} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}(t+2)$$

with $\mathbf{w}^{\mathrm{T}} = (x, y)$. By diagonalizing this matrix equation, or otherwise, convert it into independent differential equations in $x(t)$ and $y(t)$. [8]

(iii) Using the results in part (b), or otherwise, find $x(t)$ and $y(t)$, when at $t = 0$, $x = 1$, $y = 2$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -5$, and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 8$. [6]