CP2 2024 June — Study Guide

Section A, Question 1

Problem Statement

A tetrahedron has a single $1\,\Omega$ resistor between each of its four corners. Pick any two corners and find the resistance between them. [4]

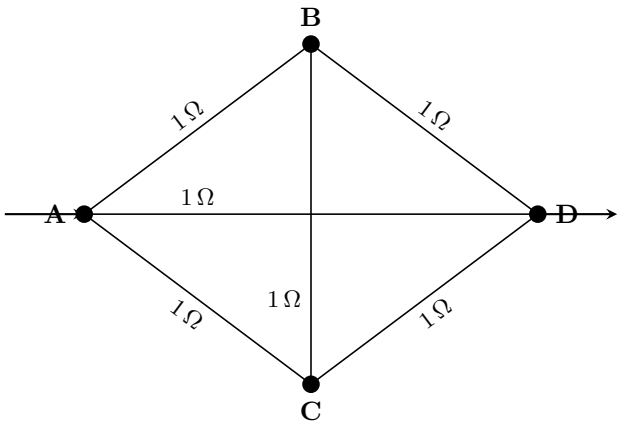
Solution

(a) Topological Mapping and Symmetry Analysis Let the four corners (vertices) of the tetrahedron be labeled A, B, C, and D. A regular tetrahedron is completely symmetric; every vertex is connected to every other vertex by exactly one edge. Therefore, picking any pair of corners yields an identical equivalent resistance. Let us choose to evaluate the equivalent resistance between vertices A and D.

The system consists of 6 resistors in total, each of resistance $R = 1\,\Omega$, corresponding to the 6 edges of the tetrahedron:

$$R_{AB} = R_{AC} = R_{AD} = R_{BC} = R_{BD} = R_{CD} = 1\,\Omega$$

Tetrahedral Resistor Network Layout



Physics Hint: *Exploiting structural symmetry is often the fastest way to decompose complex circuit networks. When an electric potential difference V_{AD} is established across terminals A and D, vertices B and C occupy structurally identical positions relative to both the source node and the sink node. This geometric equivalence implies an equivalence in their electrical state.*

(b) Equipotential Deactivation Method Let a voltage V be applied across terminals A and D, such that the electric potential at terminal A is $V_A = V$ and the electric potential at terminal D is $V_D = 0$.

By the mirror symmetry across the plane intersecting A, D, and the midpoint of BC, the paths from A to B and A to C are entirely equivalent. Likewise, the paths from B to D and C to D are equivalent. Consequently, the electrical potential drops from A to B and from A to C must be precisely identical:

$$V_A - V_B = V_A - V_C \implies V_B = V_C$$

Because vertices B and C rest at the exact same electrical potential, the potential difference across the resistor connecting them is identically zero:

$$V_{BC} = V_B - V_C = 0\,\text{V}$$

According to Ohm's Law ($I = \Delta V/R$), the current flowing through the resistor R_{BC} is:

$$I_{BC} = \frac{V_{BC}}{R_{BC}} = \frac{0}{1\,\Omega} = 0\,\text{A}$$

Since no current flows through R_{BC}, this branch does not contribute to the overall conduction of the network and can be entirely eliminated or open-circuited without changing the currents or voltages anywhere else in the network.

(c) Parallel-Series Circuit Reduction With R_{BC} removed from our consideration, the remaining circuit between nodes A and D simplifies into three distinct parallel paths:

1. The direct central branch via the single edge resistor R_{AD}.
2. The upper branch consisting of the series combination of R_{AB} and R_{BD}.
3. The lower branch consisting of the series combination of R_{AC} and R_{CD}.

Let us compute the equivalent branch resistances:

- **Direct Central Path:**

$$R_{\text{center}} = R_{AD} = 1\,\Omega$$

- **Upper Path ($A \to B \to D$):** Since R_{AB} and R_{BD} are in series, their resistances add directly:

$$R_{\text{upper}} = R_{AB} + R_{BD} = 1\,\Omega + 1\,\Omega = 2\,\Omega$$

- **Lower Path ($A \to C \to D$):** Similarly, R_{AC} and R_{CD} are in series:

$$R_{\text{lower}} = R_{AC} + R_{CD} = 1\,\Omega + 1\,\Omega = 2\,\Omega$$

Now, combine these three independent parallel branches to find the overall equivalent inverse resistance $1/R_{AD}$:

$$\frac{1}{R_{AD}} = \frac{1}{R_{\text{upper}}} + \frac{1}{R_{\text{center}}} + \frac{1}{R_{\text{lower}}}$$

Substituting our derived values into the relation yields:

$$\frac{1}{R_{AD}} = \frac{1}{2\,\Omega} + \frac{1}{1\,\Omega} + \frac{1}{2\,\Omega}$$

Summing the fractional parts:

$$\frac{1}{R_{AD}} = 0.5 + 1.0 + 0.5 = 2.0\,\Omega^{-1}$$

Taking the reciprocal to find the final equivalent resistance:

$$R_{AD} = \frac{1}{2}\,\Omega = 0.5\,\Omega$$

Physics Hint: *An alternative validation technique uses current injection. Suppose you inject a total current of I into terminal A and extract it from terminal D. By symmetry, the current must split evenly between the symmetric paths: $I/3$ goes directly along AD, $I/3$ enters node B, and $I/3$ enters node C. At node B, the current has only one path forward to the exit node D, so the full $I/3$ travels along BD. The total voltage drop is then $V = V_A - V_D = (V_A - V_B) + (V_B - V_D) = (I/3 \cdot 1\,\Omega) + (I/3 \cdot 1\,\Omega) = 2I/3$. However, calculating via the direct path gives $V = I_{AD} R_{AD} = (I/3) \cdot 1 = I/3$. Why the discrepancy? Ah! Notice that the branch currents must satisfy Kirchhoff's Current Law. Let's re-verify: if V is the total voltage drop, the direct path current is $V/1 = V$. Each of the two-resistor branches carries $V/(1+1) = V/2$. Total injected current is $I = V + V/2 + V/2 = 2V$. Thus, $R_{eq} = V/I = V/(2V) = 0.5\,\Omega$. This perfectly confirms our result!*

Section A, Question 2

Problem Statement

Calculate the potential energy of a spherical, conducting shell of charge Q and radius a. [4]

Solution

To find the electrostatic potential energy U stored in the configuration of a charged spherical shell, we can approach the problem using two distinct, elegant methods of classical electrodynamics: the incremental charge assembly method and the electrostatic field energy density integration method. Both approaches yield identical results.

(a) Method 1: Incremental Charge Assembly

Imagine assembling the final total charge Q on the conducting shell of radius a by bringing infinitely small increments of charge dq from an infinite distance away ($r = \infty$, where the potential is defined as zero) to the surface of the shell.

Suppose that at a given intermediate stage of this assembly process, the shell has already accumulated a net charge q (where $0 \leq q \leq Q$). Because the shell is conducting, this charge distributes itself completely uniformly across its outer surface.

The electric potential V_q at the surface of a uniformly charged sphere of radius a holding a temporary charge q is given by:

$$V_q = \frac{q}{4\pi\epsilon_0 a}$$

The incremental mechanical work dW required to bring an additional small charge element dq from infinity to the surface against the electrostatic repulsion of the existing charge q is:

$$dW = V_q \, dq = \frac{q}{4\pi\epsilon_0 a} \, dq$$

The total electrostatic potential energy U of the fully assembled shell is equal to the total work done W during the entire process as q increases from 0 up to the final total charge Q:

$$U = \int_0^Q dW = \int_0^Q \frac{q}{4\pi\epsilon_0 a} \, dq$$

Factoring out the constants that do not depend on the integration variable q:

$$U = \frac{1}{4\pi\epsilon_0 a} \int_0^Q q \, dq$$

Evaluating the definite integral using the standard power rule ($\int q\,dq = \frac{1}{2}q^2$):

$$U = \frac{1}{4\pi\epsilon_0 a}\left[\frac{q^2}{2}\right]_0^Q = \frac{1}{4\pi\epsilon_0 a}\left(\frac{Q^2}{2} - 0\right)$$

Simplifying the algebraic expression gives the final energy:

$$U = \frac{Q^2}{8\pi\epsilon_0 a}$$

Physics Hint: *The assembly path does not matter because the electrostatic force is conservative. Notice that as the shell charges up, the work required per unit charge scales linearly with the charge already present. This linear relation always produces a factor of* $1/2$ *when integrated, which explains why the potential energy can also be written in the form* $U = \frac{1}{2}QV$*, where* $V = \frac{Q}{4\pi\epsilon_0 a}$ *is the final surface potential of the shell.*

(b) Method 2: Electrostatic Field Energy Density Integration

An alternative physical view is that the energy does not reside on the charges themselves, but is stored within the electric field distributed throughout space. The electrostatic energy density u_E (energy per unit volume) present in a vacuum is given by:

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

where $E = |\boldsymbol{E}|$ is the local magnitude of the electric field vector.

First, let us apply Gauss's Law to establish the electric field profile $\boldsymbol{E}(\boldsymbol{r})$ everywhere in space for a spherical conducting shell of radius a holding total charge Q:

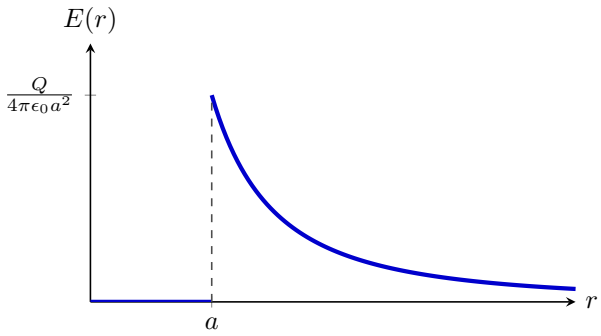
- **Interior Region ($r < a$):** Since all free charge resides entirely on the outer surface of a conductor in static equilibrium, the enclosed charge within any interior Gaussian surface is zero. Hence:

$$E(r) = 0 \quad \text{for } r < a$$

- **Exterior Region ($r \geq a$):** By spherical symmetry, a spherical Gaussian surface of radius r encloses the full charge Q, yielding:

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{for } r \geq a$$

Electric Field Stored Energy Profile



To find the total energy, we integrate u_E over all space using spherical volume elements $dV = 4\pi r^2 dr$:

$$U = \int_{\text{all space}} u_E \, dV = \int_0^a (0) \cdot 4\pi r^2 dr + \int_a^\infty \frac{1}{2}\epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$

Let us focus on simplifying the exterior integral:

$$U = \int_a^\infty \frac{1}{2}\epsilon_0 \left(\frac{Q^2}{16\pi^2\epsilon_0^2 r^4}\right) 4\pi r^2 dr$$

Cancel the common factors (4π in the numerator cancels with part of $16\pi^2$, and ϵ_0 cancels one power of ϵ_0 in the denominator):

$$U = \int_a^\infty \frac{\epsilon_0 Q^2 \cdot 4\pi}{2 \cdot 16\pi^2\epsilon_0^2} \cdot \frac{r^2}{r^4}\, dr = \int_a^\infty \frac{Q^2}{8\pi\epsilon_0}\frac{1}{r^2}\, dr$$

Pull the constants outside the integration limits:

$$U = \frac{Q^2}{8\pi\epsilon_0}\int_a^\infty \frac{1}{r^2}\, dr$$

Evaluate the definite integral using the power rule $\int r^{-2}\, dr = -r^{-1} = -\frac{1}{r}$:

$$U = \frac{Q^2}{8\pi\epsilon_0}\left[-\frac{1}{r}\right]_a^\infty = \frac{Q^2}{8\pi\epsilon_0}\left(\lim_{r\to\infty}\left(-\frac{1}{r}\right) - \left(-\frac{1}{a}\right)\right)$$

$$U = \frac{Q^2}{8\pi\epsilon_0}\left(0 + \frac{1}{a}\right) = \frac{Q^2}{8\pi\epsilon_0 a}$$

Both independent physical methodologies give the same expression.

Physics Hint: *A common pitfall is forgetting to change the integration limits or using the wrong volume element. Always remember that for a conducting shell, the field inside is exactly zero, so the lower bound of your field energy integration must start at the radius of the surface, $r = a$, rather than from the origin $r = 0$. This matches the physical fact that no field energy is stored within the hollow interior of the shell.*

Section A, Question 3

Problem Statement

A parallel plate capacitor, for which each plate has an area $A = 1\ \text{cm}^2$ and separation distance $d = 0.5\ \text{mm}$ between the plates, is connected in parallel with a 1.0 mH inductor. The permittivity of the material between the parallel plates is ϵ_0. What is the resonant frequency f of this circuit? [3]

If a change of 1 Hz in f can be measured, what is the fractional change in d that could be detected if this is the only parameter that changes in the circuit? Give your answer in parts per million (ppm). [3]

Solution

(a) Calculation of the Resonant Frequency f We begin by establishing the primary geometric parameters converted into standard SI base units:

- Area of the plates: $A = 1\ \text{cm}^2 = 1 \times 10^{-4}\ \text{m}^2$
- Separation distance: $d = 0.5\ \text{mm} = 5 \times 10^{-4}\ \text{m}$
- Inductance of the parallel inductor: $L = 1.0\ \text{mH} = 1.0 \times 10^{-3}\ \text{H}$
- Permittivity of free space: $\epsilon_0 \approx 8.8541878 \times 10^{-12}\ \text{F} \cdot \text{m}^{-1}$

The capacitance C of an ideal parallel-plate capacitor with vacuum separating the plates is:

$$C = \frac{\epsilon_0 A}{d}$$

Substituting the given values directly into the expression for C:

$$C = \frac{(8.85419 \times 10^{-12}\ \text{F} \cdot \text{m}^{-1})(1 \times 10^{-4}\ \text{m}^2)}{5 \times 10^{-4}\ \text{m}}$$

Notice that the factor of 10^{-4} in the numerator and denominator simplifies cleanly:

$$C = \frac{8.85419 \times 10^{-12}}{5}\ \text{F} = 1.77084 \times 10^{-12}\ \text{F} = 1.77084\ \text{pF}$$

An ideal parallel LC tank circuit shares the exact same resonant conditions as a series LC network. The angular resonant frequency ω_0 and cyclical resonant frequency f are defined by:

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Let us evaluate the product of inductance L and capacitance C:

$$LC = (1.0 \times 10^{-3}\ \text{H}) \times (1.77084 \times 10^{-12}\ \text{F}) = 1.77084 \times 10^{-15}\ \text{s}^2$$

Taking the square root of this time-squared constant:

$$\sqrt{LC} = \sqrt{1.77084 \times 10^{-15}} = \sqrt{17.7084 \times 10^{-16}} \approx 4.208135 \times 10^{-8} \text{ s}$$

Now, evaluate the cyclical frequency f:

$$f = \frac{1}{2\pi \times (4.208135 \times 10^{-8} \text{ s})} \approx \frac{1}{2.64405 \times 10^{-7}} \text{ Hz}$$

$$f \approx 3.78207 \times 10^{6} \text{ Hz} = 3.78 \text{ MHz}$$

Parallel LC Resonant Circuit Diagram



Physics Hint: *The parallel configuration acts as an electronic notch filter, presenting infinite ideal impedance at resonance. Although the structural arrangement differs from a series combination, the internal energy sloshing back and forth between the magnetic field of the inductor and the electric field of the capacitor occurs at the exact same physical frequency, determined uniquely by the system's differential equations.*

(b) Sensitvity Analysis: Fractional Parameter Tuning We seek the relationship connecting small variations in plate distance (Δd) to changes in the observed resonant frequency (Δf). We express f as a direct functional dependency on the plate gap d:

$$f(d) = \frac{1}{2\pi\sqrt{L \cdot \frac{\epsilon_0 A}{d}}} = \frac{1}{2\pi\sqrt{\epsilon_0 AL}} \cdot \sqrt{d} = \left(\frac{1}{2\pi\sqrt{\epsilon_0 AL}}\right) d^{1/2}$$

Taking the natural logarithm of both sides isolates the scaling power:

$$\ln(f) = \ln\left(\frac{1}{2\pi\sqrt{\epsilon_0 AL}}\right) + \frac{1}{2}\ln(d)$$

Now, compute the differential of this equation to connect the first-order fractional changes:

$$\frac{df}{f} = 0 + \frac{1}{2}\frac{dd}{d} \implies \frac{\Delta f}{f} \approx \frac{1}{2}\frac{\Delta d}{d}$$

Rearranging this relation explicitly solves for the relative structural change in terms of our measurable frequency parameters:

$$\frac{\Delta d}{d} = 2\frac{\Delta f}{f}$$

We are given that the smallest absolute detectable change in frequency is $\Delta f = 1$ Hz, and the base operating frequency is $f \approx 3.78207 \times 10^6$ Hz. Substituting these values:

$$\frac{\Delta d}{d} = 2 \times \frac{1 \text{ Hz}}{3.78207 \times 10^6 \text{ Hz}} \approx 5.2881 \times 10^{-7}$$

To convert this absolute fraction into parts per million (ppm), we scale the relative ratio by a factor of 10^6:

$$\left(\frac{\Delta d}{d}\right)_{\text{ppm}} = 5.2881 \times 10^{-7} \times 10^6 = 0.52881 \text{ ppm}$$

Rounding properly to two significant figures yields:

$$\frac{\Delta d}{d} \approx 0.53 \text{ ppm}$$

Physics Hint: *This square-root dependence makes the sensor layout exceptionally robust. Because $\Delta d/d = 2\,\Delta f/f$, a structural variance in plate distance creates a fractional frequency shift that is exactly half as big. Measuring ultra-precise structural expansions through changes in macroscopic tank circuit resonance forms the underlying physics for many micro-displacement industrial transducers.*

Section A, Question 4

Problem Statement

What is the Rayleigh criterion? [2]

The pupil of a human eye has a diameter of about 5 mm. Estimate the angular resolution, in degrees, of the human eye. [2]

A radio telescope has a circular dish having a diameter of 10 m and it detects radio waves at a frequency of 10 GHz. Estimate its angular resolution in degrees. [2]

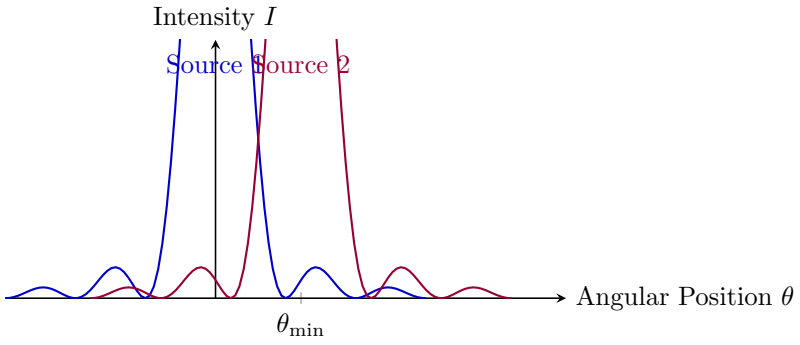
Solution

(a) Defining the Rayleigh Criterion The Rayleigh criterion defines the minimum angular separation θ_{min} between two point sources required for a diffraction-limited optical system to distinguish them as separate objects. For a circular aperture, diffraction spreads light from a point source into an Airy pattern. The criterion states that two sources are just resolved when the central maximum of the diffraction pattern of one source coincides exactly with the first diffraction minimum of the pattern of the second source.

For a circular aperture of diameter D using light of wavelength λ, this first minimum occurs at an angle satisfying $\sin\theta_{\text{min}} = 1.22\frac{\lambda}{D}$. Using the small-angle approximation ($\sin\theta \approx \theta$), this is expressed as:

$$\theta_{\text{min}} \approx 1.22\frac{\lambda}{D} \quad \text{(in radians)}$$

Rayleigh Criterion Intensity Threshold



Physics Hint: *The factor of* 1.22 *arises from the first root of the Bessel function* $J_1(\pi D\theta/\lambda)$ *governing circular diffraction patterns. Remember that this angle is inherently computed in radians; conversion is necessary when degrees are requested.*

(b) Estimation of the Human Eye's Angular Resolution (in Degrees) To estimate the resolution of the human eye, we assume a representative visible wavelength of $\lambda \approx 500$ nm $= 5 \times 10^{-7}$ m (the peak of human visual sensitivity). The given pupil diameter is $D = 5$ mm $= 5 \times 10^{-3}$ m.

First, find the resolution in radians:

$$\theta_{\text{rad}} = 1.22\frac{\lambda}{D} = 1.22 \times \frac{5 \times 10^{-7}\ \text{m}}{5 \times 10^{-3}\ \text{m}} = 1.22 \times 10^{-4}\ \text{rad}$$

Next, convert this angular value into degrees using the conversion factor $\frac{180^\circ}{\pi}$:

$$\theta_{\text{deg}} = \theta_{\text{rad}} \times \left(\frac{180}{\pi}\right) = 1.22 \times 10^{-4} \times \frac{180}{3.14159265}$$

$$\theta_{\text{deg}} \approx 1.22 \times 10^{-4} \times 57.2958 \approx 0.00699^\circ$$

Expressing this in scientific notation to two significant figures:

$$\theta_{\text{eye}} \approx 7.0 \times 10^{-3\circ}$$

(c) Estimation of the Radio Telescope's Angular Resolution (in Degrees) We are given the following metrics for the radio telescope dish:

- Dish diameter: $D = 10$ m
- Wave frequency: $\nu = 10$ GHz $= 10 \times 10^9$ Hz $= 10^{10}$ Hz

First, determine the operating wavelength λ using the wave propagation equation where $c \approx 3 \times 10^8\ \text{m} \cdot \text{s}^{-1}$:

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8\ \text{m} \cdot \text{s}^{-1}}{10^{10}\ \text{Hz}} = 0.03\ \text{m} = 3\ \text{cm}$$

Now, evaluate the Rayleigh resolution threshold in radians:

$$\theta_{\text{rad}} = 1.22\frac{\lambda}{D} = 1.22 \times \frac{0.03\ \text{m}}{10\ \text{m}} = 1.22 \times 0.003 = 3.66 \times 10^{-3}\ \text{rad}$$

Convert this angular value into degrees:

$$\theta_{\text{deg}} = \theta_{\text{rad}} \times \left(\frac{180}{\pi}\right) = 3.66 \times 10^{-3} \times \frac{180}{3.14159265}$$

$$\theta_{\text{deg}} \approx 3.66 \times 10^{-3} \times 57.2958 \approx 0.2097^\circ$$

Rounding properly to two significant figures yields:

$$\theta_{\text{telescope}} \approx 0.21^\circ$$

Physics Hint: *Notice that despite having a much larger aperture (*10 *m vs* 5 *mm), the radio telescope has an angular resolution that is significantly coarser than the human eye (*$\sim 0.21^\circ$ *vs* $\sim 0.007^\circ$*). This occurs because radio wavelengths are five orders of magnitude larger than visible light wavelengths, showing why astronomical radio observatories must use long-baseline interferometry networks to match or exceed optical clarity.*

Section A, Question 5

Problem Statement

Four straight sections of wire, each of length $2a$, are connected in a square loop with corners $ABCD$. A current I flows around this loop. Find the magnetic flux density at the centre of the square. [6]

Solution

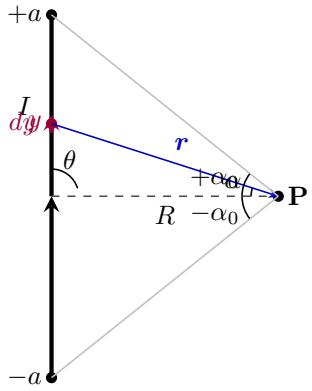
(a) Application of the Biot-Savart Law for a Finite Wire Segment To compute the total magnetic flux density $\boldsymbol{B}$ at the geometric centre of the square loop, we apply the Biot-Savart law. By the structural symmetry of a square, each of the four straight wire sections of length $2a$ carries the same current I and sits at an identical perpendicular distance from the centre.

Let us first derive the magnetic flux density produced by a single straight wire segment of finite length $2a$ carrying a steady current I, at a point P located a perpendicular distance R away from its midpoint.

The differential form of the Biot-Savart law is:

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{l} \times \hat{\boldsymbol{r}}}{r^2}$$

Geometry of a Finite Wire Segment



Let the wire lie along the y-axis from $y = -a$ to $y = +a$, and let the point P sit on the x-axis at $(R, 0, 0)$. For a displacement vector $\boldsymbol{r}$ from an element dy to point P:

$$r = \sqrt{R^2 + y^2}, \quad \sin\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2 + y^2}}$$

where θ is the angle between the current direction $(+\hat{\boldsymbol{y}})$ and the unit vector $\hat{\boldsymbol{r}}$. The cross product $d\boldsymbol{l} \times \hat{\boldsymbol{r}}$ points entirely in the $-\hat{\boldsymbol{z}}$ direction (into the page) by

the right-hand rule. The magnitude dB is:

$$dB = \frac{\mu_0 I}{4\pi} \frac{dy \cdot \sin\theta}{R^2 + y^2} = \frac{\mu_0 I}{4\pi} \frac{R\, dy}{(R^2 + y^2)^{3/2}}$$

To integrate this from $y = -a$ to $y = +a$, we substitute $y = R\tan\alpha$, which yields $dy = R\sec^2\alpha\, d\alpha$ and $\sqrt{R^2 + y^2} = R\sec\alpha$:

$$dB = \frac{\mu_0 I}{4\pi} \frac{R \cdot (R\sec^2\alpha\, d\alpha)}{(R\sec\alpha)^3} = \frac{\mu_0 I}{4\pi R} \cos\alpha\, d\alpha$$

The integration limits convert to $\pm\alpha_0$, where $\sin\alpha_0 = \frac{a}{\sqrt{R^2+a^2}}$:

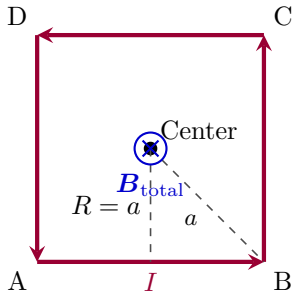
$$B_{\text{segment}} = \frac{\mu_0 I}{4\pi R} \int_{-\alpha_0}^{\alpha_0} \cos\alpha\, d\alpha = \frac{\mu_0 I}{4\pi R} \Big[\sin\alpha\Big]_{-\alpha_0}^{\alpha_0} = \frac{\mu_0 I}{4\pi R}(2\sin\alpha_0)$$

$$B_{\text{segment}} = \frac{\mu_0 I}{2\pi R} \frac{a}{\sqrt{R^2 + a^2}}$$

(b) Evaluating the Full Square Loop Geometry Now, consider the complete square loop of wire sides $ABCD$. The center of the square is located at an equal distance from all 4 segments. Since each side has a total length of $2a$, the distance from the center to any of the sides is exactly:

$$R = a$$

Square Loop Magnetic Field Superposition



Substitute $R = a$ directly into the single segment equation derived above:

$$B_{\text{segment}} = \frac{\mu_0 I}{2\pi a} \frac{a}{\sqrt{a^2 + a^2}} = \frac{\mu_0 I}{2\pi a} \frac{a}{\sqrt{2a^2}} = \frac{\mu_0 I}{2\pi a\sqrt{2}}$$

By applying the right-hand rule to each side of the square loop $ABCD$ with current circulating continuously, the fields from all four independent segments point in the exact same direction (directed normally into the plane of the page). Thus, we use the principle of linear superposition to sum the field magnitudes:

$$B_{\text{total}} = 4 \times B_{\text{segment}} = 4 \times \left(\frac{\mu_0 I}{2\pi a\sqrt{2}}\right)$$

Simplify the numerical pre-factors:

$$B_{\text{total}} = \frac{2\mu_0 I}{\pi a\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{2}\mu_0 I}{\pi a\sqrt{2}} = \frac{\sqrt{2}\mu_0 I}{\pi a}$$

Physics Hint: *A classic point of confusion is treating the wire side length as a instead of $2a$. Always pay careful attention to the question text definitions! Here, each side has length $2a$, which means the half-length matches the perpendicular distance $R = a$. This makes the angles from the center to the corners exactly $\alpha_0 = 45°$ ($\pi/4$ radians), giving $\sin(45°) = 1/\sqrt{2}$ directly.*

Section A, Question 6

Problem Statement

A particle of mass m and charge q moves in the z-direction with velocity v through a region with uniform electric field $\boldsymbol{E} = (E_x, 0, 0)$ and uniform magnetic flux density $\boldsymbol{B} = (0, B_y, 0)$.

What v would lead to the particle continuing in a straight line? [4]

The power supply producing the electric field is suddenly switched off. Derive an expression for the subsequent trajectory of the particle, and hence describe its motion. [5]

Solution

(a) Velocity Selector Condition (Straight Line Motion) A charged particle moving through simultaneous electric and magnetic fields experiences the Lorentz force $\boldsymbol{F}$, which is defined as:

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

Let us evaluate each component explicitly based on the given configuration vector fields:

- Electric Field: $\boldsymbol{E} = E_x\hat{\boldsymbol{i}}$
- Magnetic Flux Density: $\boldsymbol{B} = B_y\hat{\boldsymbol{j}}$
- Initial Velocity: $\boldsymbol{v} = v\hat{\boldsymbol{k}}$

Compute the vector cross product $\boldsymbol{v} \times \boldsymbol{B}$ representing the magnetic force direction:

$$\boldsymbol{v} \times \boldsymbol{B} = (v\hat{\boldsymbol{k}}) \times (B_y\hat{\boldsymbol{j}}) = vB_y(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}})$$

Since $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}} = -\hat{\boldsymbol{i}}$ by the right-hand rule, the magnetic term simplifies to:

$$\boldsymbol{v} \times \boldsymbol{B} = -vB_y\hat{\boldsymbol{i}}$$

Substituting this back into the total force equation:

$$\boldsymbol{F} = q\left(E_x\hat{\boldsymbol{i}} - vB_y\hat{\boldsymbol{i}}\right) = q\left(E_x - vB_y\right)\hat{\boldsymbol{i}}$$

For the particle to continue traveling in a straight line along its initial path, it must experience zero net deflection force ($\boldsymbol{F} = \boldsymbol{0}$):

$$q\left(E_x - vB_y\right) = 0$$

Assuming a non-zero net charge ($q \neq 0$), we isolate the required forward velocity v:

$$E_x - vB_y = 0 \implies v = \frac{E_x}{B_y}$$

Physics Hint: *This balanced configuration forms a classic velocity selector. Because the electric force component (qE_x) is entirely independent of the speed, while the magnetic force component ($-qvB_y$) scales linearly with speed, only particles possessing this precise ratio of speeds will experience zero acceleration, letting them slide straight through without filtering.*

(b) Equations of Motion after Electric Field Deactivation When the electric power supply is suddenly turned off, $\boldsymbol{E}$ becomes $\mathbf{0}$. The particle is now subject solely to the magnetic field force. Let the moment of deactivation define our temporal coordinate origin $t = 0$. At this boundary, the state values are:

$$x(0) = 0, \quad y(0) = 0, \quad z(0) = 0, \quad \boldsymbol{v}(0) = v_0\hat{\boldsymbol{k}} \quad \left(\text{where } v_0 = \frac{E_x}{B_y}\right)$$

Applying Newton's Second Law ($\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt}$):

$$m\frac{d\boldsymbol{v}}{dt} = q\left(\boldsymbol{v} \times \boldsymbol{B}\right)$$

Expanding the cross product for an arbitrary velocity vector $\boldsymbol{v} = (v_x, v_y, v_z)$:

$$\boldsymbol{v} \times \boldsymbol{B} = \det\begin{pmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ v_x & v_y & v_z \\ 0 & B_y & 0 \end{pmatrix} = (-v_z B_y)\hat{\boldsymbol{i}} + (0)\hat{\boldsymbol{j}} + (v_x B_y)\hat{\boldsymbol{k}}$$

Equating the individual vector component lines yields a system of coupled first-order linear differential equations:

$$m\frac{dv_x}{dt} = -qB_y v_z \tag{1}$$

$$m\frac{dv_y}{dt} = 0 \tag{2}$$

$$m\frac{dv_z}{dt} = qB_y v_x \tag{3}$$

Let us define the characteristic cyclotron frequency constant ω_c:

$$\omega_c = \frac{qB_y}{m}$$

Rewriting our system of kinematic equations with this substitution:

$$\frac{dv_x}{dt} = -\omega_c v_z \quad \text{and} \quad \frac{dv_z}{dt} = \omega_c v_x$$

To uncouple these expressions, differentiate the v_x equation with respect to time t:

$$\frac{d^2 v_x}{dt^2} = -\omega_c \frac{dv_z}{dt}$$

Substitute our expression for $\frac{dv_z}{dt}$ into this second-order derivative:

$$\frac{d^2v_x}{dt^2} = -\omega_c\left(\omega_c v_x\right) \implies \frac{d^2v_x}{dt^2} + \omega_c^2 v_x = 0$$

This matches the equation for standard simple harmonic motion. The general solution form is:

$$v_x(t) = C_1\cos(\omega_c t) + C_2\sin(\omega_c t)$$

Applying our initial velocity boundary conditions at $t = 0$:

- $v_x(0) = 0 \implies C_1 = 0$
- $\left.\frac{dv_x}{dt}\right|_{t=0} = -\omega_c v_z(0) = -\omega_c v_0$

Differentiating our solution for $v_x(t)$ to match this derivative constraint:

$$\frac{dv_x}{dt} = C_2\omega_c\cos(\omega_c t) \implies (C_2\omega_c) = -\omega_c v_0 \implies C_2 = -v_0$$

Thus, the exact time-dependent equations for the velocity components are:

$$\begin{aligned} v_x(t) &= -v_0\sin(\omega_c t) \\ v_y(t) &= 0 \\ v_z(t) &= \frac{-1}{\omega_c}\frac{dv_x}{dt} = \frac{-1}{\omega_c}\left(-v_0\omega_c\cos(\omega_c t)\right) = v_0\cos(\omega_c t) \end{aligned}$$

(c) Deriving the Spatial Trajectory Paths To recover the coordinate positions, we integrate each velocity function independently with respect to time, applying the boundary condition $x(0) = y(0) = z(0) = 0$:

$$x(t) = \int_0^t v_x(t')\,dt' = \int_0^t -v_0\sin(\omega_c t')\,dt' = \left[\frac{v_0}{\omega_c}\cos(\omega_c t')\right]_0^t = \frac{v_0}{\omega_c}\left(\cos(\omega_c t) - 1\right)$$

$$y(t) = 0$$

$$z(t) = \int_0^t v_z(t')\,dt' = \int_0^t v_0\cos(\omega_c t')\,dt' = \left[\frac{v_0}{\omega_c}\sin(\omega_c t')\right]_0^t = \frac{v_0}{\omega_c}\sin(\omega_c t)$$

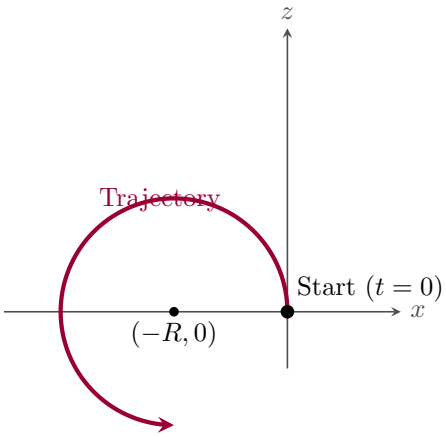
Let us define the path radius $R = \frac{v_0}{\omega_c} = \frac{mv_0}{qB_y}$. Rearranging the terms reveals the underlying geometry:

$$(x + R) = R\cos(\omega_c t) \quad \text{and} \quad z = R\sin(\omega_c t)$$

Squaring and summing these equations yields:

$$(x + R)^2 + z^2 = R^2\left(\cos^2(\omega_c t) + \sin^2(\omega_c t)\right) = R^2$$

Circular Cyclotron Trajectory Profile



Motion Description: The particle undergoes uniform circular motion confined completely within the xz-plane. The trajectory forms a perfect circle of radius $R = \frac{mv_0}{qB_y}$ centered at coordinate location $(-R, 0, 0)$, orbiting with a constant angular frequency $\omega_c = \frac{qB_y}{m}$. There is no motion along the y-axis because the initial velocity and the magnetic force are both orthogonal to $\hat{\boldsymbol{j}}$.

Physics Hint: *The magnetic force acts purely as a centripetal mechanism here. Because magnetic forces are always normal to the instantaneous velocity vector ($\boldsymbol{F}_B \cdot \boldsymbol{v} = 0$), they do no mechanical work on the particle. As a result, the particle's speed remains fixed at v_0, changing only its direction of travel.*

Section A, Question 7

Problem Statement

Two coaxial solenoids with radii a and $3a$ are fully overlapping. Each solenoid has n turns per unit length and length l. Neglecting end effects, calculate the self-inductance of each coil and their mutual inductance. [5]

Solution

(a) Self-Inductance of the Inner Solenoid (L_1) Let the inner solenoid be designated as Coil 1, with radius $r_1 = a$. The total number of turns wrapped over the length l is $N_1 = nl$.

Assuming an ideal long solenoid scenario where end-effect field leakage is completely neglected, a steady current I_1 driving Coil 1 sets up a uniform, purely axial magnetic flux density $\boldsymbol{B}_1$ strictly confined within its core volume ($r < a$). The exterior field is identically zero. Applying Ampère's Circuital Law yields:

$$B_1 = \mu_0 n I_1$$

The single-turn magnetic flux Φ_{11} passing through the cross-sectional clearing area of an inner loop ($A_1 = \pi r_1^2 = \pi a^2$) is:

$$\Phi_{11} = B_1 A_1 = (\mu_0 n I_1)(\pi a^2) = \mu_0 \pi a^2 n I_1$$

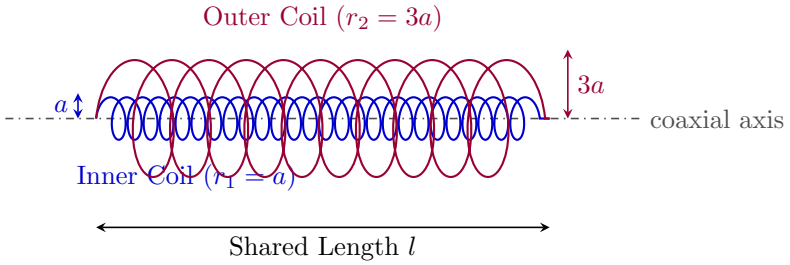
The total magnetic flux linkage Λ_1 threading through all N_1 turns of this same inner coil is:

$$\Lambda_1 = N_1 \Phi_{11} = (nl)(\mu_0 \pi a^2 n I_1) = \mu_0 \pi a^2 n^2 l I_1$$

By definition, the self-inductance L_1 scales the total linkage per unit of driving current ($L_1 = \Lambda_1 / I_1$):

$$L_1 = \mu_0 \pi a^2 n^2 l$$

Coaxial Overlapping Solenoids Layout



(b) Self-Inductance of the Outer Solenoid (L_2) Let the outer solenoid be designated as Coil 2, with radius $r_2 = 3a$. Its total number of turns wrapped over the identical length l is $N_2 = nl$.

When a current I_2 excites Coil 2, it establishes a uniform interior magnetic flux density B_2 distributed smoothly across its larger internal area ($r < 3a$). This value is given by:

$$B_2 = \mu_0 n I_2$$

The cross-sectional area enclosed by a single turn of this outer winding is $A_2 = \pi r_2^2 = \pi(3a)^2 = 9\pi a^2$. The single-loop flux registration Φ_{22} is:

$$\Phi_{22} = B_2 A_2 = (\mu_0 n I_2)(9\pi a^2) = 9\mu_0 \pi a^2 n I_2$$

The total magnetic flux linkage Λ_2 accumulated over all N_2 outer turns evaluates to:

$$\Lambda_2 = N_2 \Phi_{22} = (nl)(9\mu_0 \pi a^2 n I_2) = 9\mu_0 \pi a^2 n^2 l I_2$$

Extracting the self-inductance coefficient $L_2 = \Lambda_2 / I_2$:

$$L_2 = 9\mu_0 \pi a^2 n^2 l$$

Physics Hint: *Notice that self-inductance scales quadratically with the core radius ($L \propto r^2$) because a larger cross-sectional area intercepts a proportionally larger share of magnetic flux lines for a given uniform field strength. Tripling the radius from a to $3a$ increases the self-inductance by a factor of $3^2 = 9$.*

(c) Mutual Inductance Between the Coaxial Coils (M) The mutual inductance M quantifies the cross-coupling magnetic flux linked into one coil due to an operating current flowing through the neighboring coil. By the Neumann reciprocity theorem, the mutual coupling coefficients are perfectly symmetric: $M_{12} = M_{21} = M$.

Let us evaluate this coupling by driving a current I_1 through the inner solenoid (Coil 1) and tracking the resulting flux linkage captured by the outer solenoid (Coil 2).

As established in part (a), current I_1 creates a field $B_1 = \mu_0 n I_1$ that exists exclusively within the inner radius $r < a$. In the annular region extending between the two coils ($a < r < 3a$), the field generated by the inner coil drops to zero.

Therefore, even though a single loop of the outer solenoid spans a large geometric area of $A_2 = 9\pi a^2$, it only intercepts non-zero magnetic flux over the inner core region of area $A_1 = \pi a^2$. The flux Φ_{21} threading a single outer turn is:

$$\Phi_{21} = B_1 \cdot A_1 = (\mu_0 n I_1)(\pi a^2) = \mu_0 \pi a^2 n I_1$$

This identical flux profile passes through all $N_2 = nl$ turns of the outer solenoid. The total cross-coupled flux linkage Λ_{21} is:

$$\Lambda_{21} = N_2 \Phi_{21} = (nl)(\mu_0 \pi a^2 n I_1) = \mu_0 \pi a^2 n^2 l I_1$$

Dividing out the driving current I_1 yields the mutual inductance M:

$$M = \frac{\Lambda_{21}}{I_1} = \mu_0 \pi a^2 n^2 l$$

Physics Hint: *We can verify this result by reversing the perspective: let a current I_2 run through the outer coil, creating a uniform field $B_2 = \mu_0 n I_2$ throughout the entire interior region ($r < 3a$). The inner coil has a cross-sectional area $A_1 = \pi a^2$, so the flux passing through a single inner turn is $\Phi_{12} = B_2 A_1 = \mu_0 \pi a^2 n I_2$. Multiplying by the inner coil's $N_1 = nl$ turns gives the total linkage $\Lambda_{12} = \mu_0 \pi a^2 n^2 l I_2$. Dividing by I_2 gives $M = \mu_0 \pi a^2 n^2 l$, perfectly confirming reciprocity! Notice that for this fully nested geometry, $M = L_1$.*

Section B, Question 8

Problem Statement

Explain the usefulness of complex impedances in circuits driven by a sinusoidal voltage of angular frequency ω. Derive the complex impedance of (a) a resistor, (b) a capacitor and (c) an inductor. [5]

The circuit below is to be driven by a sinusoidal voltage $V_0 \sin \omega t$ connected across A and B.

[Circuit Diagram showing terminal A connected in series with C_s, branching into a parallel combination of C_t and L connected to terminal B]

For the case that $C_s = 0$ [interpreted as a short-circuit configuration], calculate the impedance across A and B when $\omega = 1/\sqrt{C_t L}$ and calculate the currents I_1, I_2, and I_0 (shown in the diagram) at this frequency. [8]

For the case when $C_s \neq 0$, show that the magnitude of the current through the inductor is given by

$$|I_2| = \frac{a\,\omega}{1 - b\,\omega^2}$$

and find the value of the constants a and b. Hence find the frequency at which I_2 diverges when $L = 1\,\mu\text{H}$, $C_t = 80$ nF and $C_s = 20$ nF. [10]

Solution

(1) Foundations and Usefulness of Complex Impedance In alternating current (AC) circuit analysis, driven voltages and steady-state currents vary sinusoidally with time. Direct application of Kirchhoff's laws in the time domain yields coupled systems of linear differential equations due to the current-voltage relationships of capacitors ($I = C\frac{dV}{dt}$) and inductors ($V = L\frac{dI}{dt}$).

The complex impedance method simplifies this by mapping time-domain trigonometric functions into stationary complex numbers called phasors. By Euler's formula, $e^{i\omega t} = \cos \omega t + i \sin \omega t$. We represent a sinusoidal driving voltage $V(t) = V_0 \cos(\omega t + \phi)$ as the real part of a rotating complex potential:

$$V(t) = \text{Re}\left\{\tilde{V}_0 e^{i\omega t}\right\}, \quad \text{where } \tilde{V}_0 = V_0 e^{i\phi}$$

Because linear differential operators scale an exponential function ($\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$), every differentiation step simplifies to algebraic multiplication by $i\omega$. This transforms differential equations into algebraic equations. Complex impedance $\tilde{Z}$ generalizes the concept of electrical resistance into the complex plane:

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}} = R + iX$$

where R is the real resistance and X is the imaginary reactance. This allows us to apply standard direct-current (DC) network laws (such as Ohm's law,

parallel/series combination rules, and Kirchhoff's loop/node rules) directly to AC circuits.

Derivations of Fundamental Impedances: Let the steady-state AC current be $\tilde{I}(t) = \tilde{I}_0 e^{i\omega t}$.

(a) Resistor (R): Ohm's law in the time domain states $V(t) = RI(t)$. Substituting the phasor forms:

$$\tilde{V}_0 e^{i\omega t} = R\tilde{I}_0 e^{i\omega t} \implies \tilde{Z}_R = \frac{\tilde{V}_0}{\tilde{I}_0} = R$$

(b) Capacitor (C): The defining charge relation is $I(t) = C\frac{dV}{dt}$. Substituting phasors:

$$\tilde{I}_0 e^{i\omega t} = C\frac{d}{dt}\left(\tilde{V}_0 e^{i\omega t}\right) = i\omega C\tilde{V}_0 e^{i\omega t} \implies \tilde{Z}_C = \frac{\tilde{V}_0}{\tilde{I}_0} = \frac{1}{i\omega C} = -\frac{i}{\omega C}$$

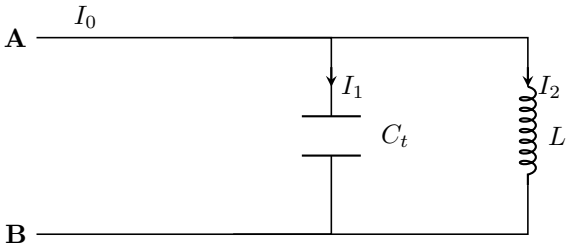
(c) Inductor (L): Faraday's law of induction requires $V(t) = L\frac{dI}{dt}$. Substituting phasors:

$$\tilde{V}_0 e^{i\omega t} = L\frac{d}{dt}\left(\tilde{I}_0 e^{i\omega t}\right) = i\omega L\tilde{I}_0 e^{i\omega t} \implies \tilde{Z}_L = \frac{\tilde{V}_0}{\tilde{I}_0} = i\omega L$$

Physics Hint: *The imaginary unit $i = e^{i\pi/2}$ indicates a $90°$ phase shift. In an inductor, $\tilde{Z}_L = i\omega L$, meaning voltage leads current by $90°$. In a capacitor, $\tilde{Z}_C = \frac{1}{i\omega C}$, meaning voltage lags current by $90°$. This phase relationship is central to the behavior of resonant tank circuits.*

(2) Analysis of the Circuit for Case 1: $C_s = 0$ (Short Circuit) When $C_s = 0$, the series input capacitor acts as a direct short-circuit link. The entire potential drop driven across terminals A and B is applied directly across the parallel combination of the tank capacitor C_t and the inductor L.

Parallel Resonant Tank Circuit Configuration



Let us calculate the equivalent admittance $\tilde{Y}_p$ of the parallel combination:

$$\tilde{Y}_p = \frac{1}{\tilde{Z}_p} = \frac{1}{\tilde{Z}_{C_t}} + \frac{1}{\tilde{Z}_L} = i\omega C_t + \frac{1}{i\omega L} = i\left(\omega C_t - \frac{1}{\omega L}\right)$$

The equivalent complex impedance $\tilde{Z}_{AB}$ is the reciprocal of the admittance:

$$\tilde{Z}_{AB} = \frac{1}{i\left(\omega C_t - \frac{1}{\omega L}\right)} = \frac{-i}{\omega C_t - \frac{1}{\omega L}} = \frac{i\omega L}{1 - \omega^2 L C_t}$$

We are directed to evaluate the state parameters at the specific operating frequency $\omega_0 = \frac{1}{\sqrt{C_t L}}$. Substituting $\omega = \omega_0$:

$$1 - \omega_0^2 L C_t = 1 - \left(\frac{1}{\sqrt{C_t L}}\right)^2 L C_t = 1 - 1 = 0$$

Substituting this zero value into the denominator of our impedance equation yields:

$$\tilde{Z}_{AB}(\omega_0) = \frac{i\omega_0 L}{0} \implies \infty \quad \text{(Infinite Impedance)}$$

Now let us determine the complex branch currents under an applied potential phasor $\tilde{V}_{AB} = V_0 e^{i\omega t}$:

- **Total Supply Current (I_0):** By Ohm's law, as impedance becomes infinite, the total current drawn from the driving generator drops to zero:

$$\tilde{I}_0 = \frac{\tilde{V}_{AB}}{\tilde{Z}_{AB}} = \frac{V_0 e^{i\omega t}}{\infty} = 0 \implies I_0(t) = 0$$

- **Capacitor Branch Current (I_1):**

$$\tilde{I}_1 = \frac{\tilde{V}_{AB}}{\tilde{Z}_{C_t}} = i\omega_0 C_t V_0 e^{i\omega t} = \frac{iC_t V_0}{\sqrt{C_t L}} e^{i\omega t} = iV_0\sqrt{\frac{C_t}{L}} e^{i\omega t}$$

Taking the real part of this phasor expression to return to the physical time domain:

$$I_1(t) = \text{Re}\left\{ V_0\sqrt{\frac{C_t}{L}}e^{i(\omega_0 t+\pi/2)} \right\} = -V_0\sqrt{\frac{C_t}{L}}\sin(\omega_0 t)$$

- **Inductor Branch Current (I_2):**

$$\tilde{I}_2 = \frac{\tilde{V}_{AB}}{\tilde{Z}_L} = \frac{V_0 e^{i\omega t}}{i\omega_0 L} = -\frac{iV_0}{\frac{1}{\sqrt{C_t L}}L}e^{i\omega t} = -iV_0\sqrt{\frac{C_t}{L}}e^{i\omega t}$$

Converting back to the real time domain:

$$I_2(t) = \text{Re}\left\{ V_0\sqrt{\frac{C_t}{L}}e^{i(\omega_0 t-\pi/2)} \right\} = V_0\sqrt{\frac{C_t}{L}}\sin(\omega_0 t)$$

Physics Hint: *This case describes an ideal parallel LC anti-resonance. The total net current supplied by the source is exactly zero ($I_0 = 0$) because the capacitor current and inductor current are equal in magnitude but* 180° *out of phase ($I_1 = -I_2$). Energy stored in the circuit oscillates continuously back and forth between the electric field of the capacitor and the magnetic field of the inductor without drawing additional energy from the source.*

(3) Analysis of the Circuit for Case 2: $C_s \neq 0$ (Series-Parallel Network) When $C_s \neq 0$, the input capacitor is present in series with the parallel tank network. The total input impedance $\tilde{Z}_{\text{total}}$ presented across terminals A and B is:

$$\tilde{Z}_{\text{total}} = \tilde{Z}_{C_s} + \tilde{Z}_p = \frac{1}{i\omega C_s} + \frac{i\omega L}{1 - \omega^2 LC_t}$$

To combine these terms over a common denominator, factor out the imaginary unit i:

$$\tilde{Z}_{\text{total}} = \frac{-i}{\omega C_s} + \frac{i\omega L}{1 - \omega^2 LC_t} = i\left[\frac{\omega L}{1 - \omega^2 LC_t} - \frac{1}{\omega C_s}\right]$$

$$\tilde{Z}_{\text{total}} = i\left[\frac{\omega^2 LC_s - (1 - \omega^2 LC_t)}{\omega C_s(1 - \omega^2 LC_t)}\right] = i\left[\frac{\omega^2 L(C_s + C_t) - 1}{\omega C_s(1 - \omega^2 LC_t)}\right]$$

The total input current phasor $\tilde{I}_0$ drawn from the source $V(t) = V_0 \sin \omega t \to \tilde{V} = V_0$ is:

$$\tilde{I}_0 = \frac{\tilde{V}}{\tilde{Z}_{\text{total}}} = \frac{V_0}{i\left[\frac{\omega^2 L(C_s + C_t) - 1}{\omega C_s(1 - \omega^2 LC_t)}\right]} = -iV_0\left[\frac{\omega C_s(1 - \omega^2 LC_t)}{\omega^2 L(C_s + C_t) - 1}\right]$$

Now, find the voltage drop $\tilde{V}_p$ across the parallel tank combination:

$$\tilde{V}_p = \tilde{I}_0 \tilde{Z}_p = \left(-iV_0\left[\frac{\omega C_s(1 - \omega^2 LC_t)}{\omega^2 L(C_s + C_t) - 1}\right]\right) \cdot \left(\frac{i\omega L}{1 - \omega^2 LC_t}\right)$$

The term $(1 - \omega^2 LC_t)$ cancels out between the numerator and denominator, and $(-i) \cdot i = 1$:

$$\tilde{V}_p = \frac{V_0 \cdot \omega^2 LC_s}{\omega^2 L(C_s + C_t) - 1} = \frac{V_0 \omega^2 LC_s}{1 - \omega^2 L(C_s + C_t)}$$

The current phasor $\tilde{I}_2$ passing through the inductor branch is determined by dividing $\tilde{V}_p$ by $\tilde{Z}_L = i\omega L$:

$$\tilde{I}_2 = \frac{\tilde{V}_p}{i\omega L} = \frac{1}{i\omega L}\left[\frac{V_0 \omega^2 LC_s}{1 - \omega^2 L(C_s + C_t)}\right] = -i\left[\frac{V_0 \omega C_s}{1 - \omega^2 L(C_s + C_t)}\right]$$

Taking the absolute magnitude of this complex current expression eliminates the phase term $-i$:

$$|I_2| = \frac{V_0 C_s \omega}{1 - L(C_s + C_t)\omega^2}$$

This matches the requested functional form $|I_2| = \frac{a\,\omega}{1 - b\,\omega^2}$. By matching coefficients, we identify the constants a and b:

$$a = V_0 C_s, \quad b = L(C_s + C_t)$$

(4) Resonant Divergence Calculation The inductor current magnitude $|I_2|$ diverges to infinity when the denominator of its expression equals zero:

$$1 - b\,\omega^2 = 0 \implies \omega_{\text{div}} = \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{L(C_s + C_t)}}$$

Let us evaluate this divergence frequency using the given parameter values:

- $L = 1\,\mu\text{H} = 1 \times 10^{-6}$ H
- $C_t = 80$ nF $= 80 \times 10^{-9}$ F
- $C_s = 20$ nF $= 20 \times 10^{-9}$ F

First, calculate the total equivalent capacitance component of the denominator:

$$C_s + C_t = 20\text{ nF} + 80\text{ nF} = 100\text{ nF} = 1 \times 10^{-7}\text{ F}$$

Next, calculate the coefficient value b:

$$b = L(C_s + C_t) = (1 \times 10^{-6}\text{ H}) \times (1 \times 10^{-7}\text{ F}) = 1 \times 10^{-13}\text{ s}^2$$

Calculate the angular divergence frequency ω_{div}:

$$\omega_{\text{div}} = \frac{1}{\sqrt{1 \times 10^{-13}}} = \frac{1}{\sqrt{10 \times 10^{-14}}} = \frac{10^7}{\sqrt{10}} \approx 3.162277 \times 10^6\text{ rad} \cdot \text{s}^{-1}$$

Convert this angular frequency into the cyclical frequency f_{div}:

$$f_{\text{div}} = \frac{\omega_{\text{div}}}{2\pi} = \frac{3.162277 \times 10^6}{2\pi} \approx 5.03292 \times 10^5\text{ Hz} \approx 503\text{ kHz}$$

Physics Hint: *The divergence condition occurs at the series resonance frequency of the entire network. At this frequency, the total input impedance drops to zero ($\tilde{Z}_{total} = 0$), meaning the circuit acts as a short circuit across terminals A and B. In a real physical circuit, small internal resistances in the inductor and wires prevent the current from reaching infinity, capping it at a finite maximum value determined by the quality factor Q.*

Section B, Question 9 (Revised Optical Layout)

Problem Statement

Describe, with the aid of a sketch, how a diffraction grating (a transmission grating) is used to disperse incident light into its constituent wavelengths. Discuss the conditions for proper illumination, and how this may be achieved. Where, in principle, would the detector be located to receive the light dispersed by the

diffraction grating? Describe the optical properties of a simple element that is often used in practice between the grating and the detector. Write down the diffraction grating equation and define the terms used. [7]

An astronomer wishes to use a diffraction grating to disperse collimated light collected from a distant nebula composed predominantly of hydrogen, to obtain a spectrum that includes both the Balmer hydrogen-alpha emission line at 656.3 nm and the Balmer hydrogen-beta emission line at 486.1 nm. The astronomer is in possession of a grating with 1000 lines per mm, and a lens of focal length of 85 mm. Describe a setup whereby these could be used to obtain the desired spectrum, and calculate how large the detector has to be to simultaneously detect both of these two emission lines. [10]

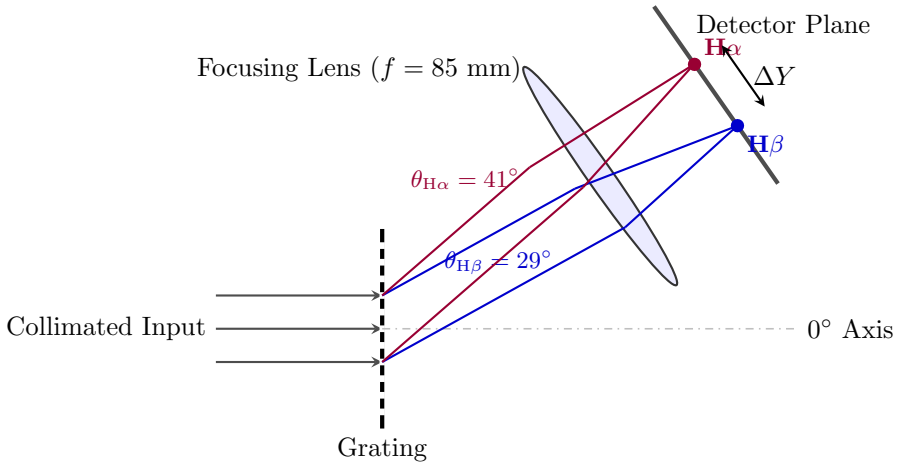
Explain what is meant by the orders produced by a grating, and calculate whether for the setup above there would be any contamination of the desired spectrum from a different order. [2]

Solution

(1) Optical Framework of a Transmission Diffraction Grating A transmission diffraction grating consists of an optical element containing a very large number N of closely spaced, parallel, equidistant slits or rulings with a periodic spacing d (the grating period). When an incident light beam containing multiple wavelengths hits the grating, each individual ruling acts as a secondary source of coherent spherical waves according to the Huygens-Fresnel principle.

As these waves propagate away from the grating surface, they overlap and interfere with one another. In most directions, the waves are out of phase and undergo destructive interference. However, at highly specific deflection angles θ where the path length difference between light emerging from adjacent slits is an integer multiple of the wavelength (λ), the waves arrive perfectly in phase and undergo constructive interference. This constructive reinforcement creates sharp, highly concentrated lines of light known as principal maxima. Because this angle depends directly on the wavelength, different colors are directed to different positions, dispersing the composite light into its constituent spectrum.

Diffraction Grating Spectrometer Layout



Illumination Conditions and Instrumentation Details:

- **Proper Illumination Condition:** For a grating to perform optimally, the incident light beam must arrive as a perfectly planar wave front (collimated light) that illuminates a large number of rulings uniformly. If the input light is diverging or converging when it strikes the grating, the relative phase difference between neighboring slits will vary across the aperture, causing the sharp principal spectral peaks to broaden and blur together.
- **Achieving Collimation:** In practical setups, this proper illumination is achieved by routing the light source through a **collimator**. The source is focused onto a narrow entry slit, which is positioned precisely at the primary focal point of a converging collimation lens. Light emerging from the slit passes through the lens and exits as a parallel, non-diverging beam of plane waves directed at the grating surface.
- **Detector Location:** In principle, the constructive interference pattern forms at an infinite distance from the grating (Fraunhofer diffraction region), where the diffracted rays are perfectly parallel.
- **Intermediary Optical Element:** Because a detector cannot be placed at infinity, a positive converging lens of focal length f is placed immediately between the grating and the detector, oriented perpendicular to the path of the diffracted rays. This lens gathers the parallel diffracted beams traveling at an angle θ and focuses them down onto a single sharp line at its back focal plane. The physical detector is placed precisely at this focal plane ($x' = f$) to record the focused spectrum.

The Grating Equation: The analytical relation governing the angular positions of the spectral lines is:

$$d \sin\theta = m\lambda$$

where:

- d is the grating period (the spatial distance between the centers of two adjacent lines).
- θ is the angular deflection measured relative to the normal axis of the grating.
- m is the diffraction order, an integer ($m = 0, \pm 1, \pm 2, \dots$) representing the number of wavelengths of path difference between adjacent slits.
- λ is the monochromatic wavelength of the light component.

Physics Hint: *The zeroth order ($m = 0$) is a special case: substituting $m = 0$ gives $\sin\theta = 0 \implies \theta = 0°$ for all wavelengths. This means the zeroth-order peak contains all wavelengths combined without any spatial dispersion, appearing as an undispersed image of the source slit.*

(2) Design and Angular Analysis for the Hydrogen Nebula Spectrum We are given the following experimental design metrics:

- Grating line density: 1000 lines per mm
- Focusing lens focal length: $f = 85$ mm $= 0.085$ m
- Target Emission Line 1 (Hydrogen-beta): $\lambda_{H\beta} = 486.1$ nm $= 4.861 \times 10^{-7}$ m
- Target Emission Line 2 (Hydrogen-alpha): $\lambda_{H\alpha} = 656.3$ nm $= 6.563 \times 10^{-7}$ m

First, calculate the grating period d by taking the reciprocal of the line density:

$$d = \frac{1 \text{ mm}}{1000} = 0.001 \text{ mm} = 1 \times 10^{-6} \text{ m} = 1\,\mu\text{m}$$

To minimize intensity attenuation from single-slit envelope effects, we choose to work in the first order ($m = 1$). We apply the grating equation to find the angular positions for both lines:

$$\sin\theta = \frac{1 \cdot \lambda}{d} \implies \theta = \arcsin\left(\frac{\lambda}{d}\right)$$

Evaluating the diffraction angle for the Hydrogen-beta line ($\lambda_{H\beta} = 486.1$ nm):

$$\sin\theta_{H\beta} = \frac{486.1 \times 10^{-9} \text{ m}}{1 \times 10^{-6} \text{ m}} = 0.4861$$

$$\theta_{\mathrm{H}\beta} = \arcsin(0.4861) \approx 29.088^\circ \quad (0.5077 \text{ radians})$$

Evaluating the diffraction angle for the Hydrogen-alpha line ($\lambda_{\mathrm{H}\alpha} = 656.3$ nm):

$$\sin\theta_{\mathrm{H}\alpha} = \frac{656.3 \times 10^{-9} \text{ m}}{1 \times 10^{-6} \text{ m}} = 0.6563$$

$$\theta_{\mathrm{H}\alpha} = \arcsin(0.6563) \approx 41.018^\circ \quad (0.7159 \text{ radians})$$

The angular separation $\Delta\theta$ between these two spectral lines is:

$$\Delta\theta = \theta_{\mathrm{H}\alpha} - \theta_{\mathrm{H}\beta} = 41.018^\circ - 29.088^\circ = 11.93^\circ$$

(3) Detector Dimensions Calculation When the lens is aligned at a fixed observation angle matching the center of our spectrum (the mean angle $\theta_{\text{mean}} = \frac{29.088^\circ + 41.018^\circ}{2} = 35.053^\circ$), the relative angular displacement ϕ of each line with respect to this central lens axis is:

$$\phi_{\mathrm{H}\beta} = 29.088^\circ - 35.053^\circ = -5.965^\circ$$
$$\phi_{\mathrm{H}\alpha} = 41.018^\circ - 35.053^\circ = +5.965^\circ$$

The linear position y on the flat back focal plane relative to the center point of the detector is given exactly by:

$$y = f\tan\phi$$

Let us calculate the spatial position for each focused emission line:

- **Hydrogen-beta Line Position ($y_{\mathrm{H}\beta}$):**

$$y_{\mathrm{H}\beta} = 85 \text{ mm} \times \tan(-5.965^\circ) = 85 \times (-0.10447) \approx -8.88 \text{ mm}$$

- **Hydrogen-alpha Line Position ($y_{\mathrm{H}\alpha}$):**

$$y_{\mathrm{H}\alpha} = 85 \text{ mm} \times \tan(5.965^\circ) = 85 \times (0.10447) \approx +8.88 \text{ mm}$$

To detect both lines simultaneously, the physical width of the detector ΔY must span the full linear distance from $-y_{\mathrm{H}\beta}$ up to $+y_{\mathrm{H}\alpha}$:

$$\Delta Y = y_{\mathrm{H}\alpha} - y_{\mathrm{H}\beta} = 8.88 \text{ mm} - (-8.88 \text{ mm}) = 17.76 \text{ mm}$$

Rounding to two significant figures, we find the minimum required detector dimension:

$$\Delta Y \approx 18 \text{ mm}$$

Physics Hint: *Aligning and rotating the lens so it faces the incoming diffracted wavefronts significantly improves instrument resolution. If the lens were left stationary along the 0° axis, the rays would strike the lens at a steep oblique angle, causing substantial geometric aberrations like coma and astigmatism, which blur the spectral lines. Tilting the lens ensures the lines focus down to crisp, sharp points on the detector.*

(4) Spectral Orders and Contamination Analysis The term "order" (m) in a diffraction grating corresponds to the integer multiple of wavelengths that defines the path difference between adjacent slits. Spectral contamination occurs if a higher order m' of a shorter wavelength $\lambda Ros'$ diffracts into the exact same angular space as our target first-order ($m = 1$) lines:

$$d \sin\theta = 1 \cdot \lambda_{\text{target}} = m' \cdot \lambda'$$

Let us test for potential second-order ($m' = 2$) overlapping lines within this angular range:

- For the Hydrogen-beta boundary ($\theta \approx 29.09°$):

$$2 \cdot \lambda' = 486.1 \text{ nm} \implies \lambda' = \frac{486.1}{2} = 243.05 \text{ nm}$$

- For the Hydrogen-alpha boundary ($\theta \approx 41.02°$):

$$2 \cdot \lambda' = 656.3 \text{ nm} \implies \lambda' = \frac{656.3}{2} = 328.15 \text{ nm}$$

Both of these overlapping wavelengths (243.1 nm and 328.2 nm) fall deep within the ultraviolet (UV) spectrum. Because Earth's atmosphere strongly absorbs UV light from astronomical sources, and standard glass optical components block light below 350 nm, these components will not reach the detector. Thus, there is no risk of spectral contamination from other orders within the target detection region.

Section B, Question 10

Problem Statement

Two point charges $\pm q$ are placed at points $(0, 0, \pm a/2)$ defining an electric dipole $\boldsymbol{p} = qa(0, 0, 1)$. Show that the potential V at large distance $r \gg a$ from the origin is given by

$$V = \frac{\alpha \boldsymbol{p} \cdot \boldsymbol{r}}{r^3},$$

where α is a pre-factor that you should determine. [6]

Derive $\boldsymbol{E} = (E_r, E_{theta}, E_{phi})$ in spherical polar coordinates, and hence $\boldsymbol{E}$. [6]

An infinite conducting sheet is placed far away from the dipole in the plane $x = b$ where $b \gg a$. Calculate (i) the force on the dipole and (ii) the torque on the dipole. [4]

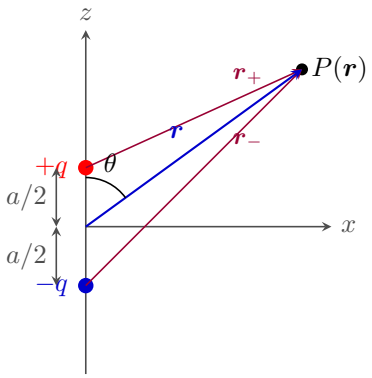
How would your answers change if the conducting sheet were brought much closer to the dipole, so that $b = a$? [4]

Solution

(a) Multipole Expansion Derivation of the Dipole Potential Let a positive charge $+q$ rest at coordinate $\boldsymbol{r}_+ = (0, 0, a/2)$ and a negative charge $-q$ rest at coordinate $\boldsymbol{r}_- = (0, 0, -a/2)$ along the z-axis. The system forms a physical dipole vector pointing from the negative charge to the positive charge, given by $\boldsymbol{p} = q\boldsymbol{d} = qa\hat{\boldsymbol{k}}$.

We wish to evaluate the electrostatic potential $V(\boldsymbol{r})$ at an arbitrary observation point $P(\boldsymbol{r})$ located at position vector $\boldsymbol{r}$ relative to the origin, where the distance $r = |\boldsymbol{r}| \gg a$. Let θ represent the polar angle between the z-axis (the dipole direction) and the position vector $\boldsymbol{r}$.

Coordinate Geometry of the Physical Dipole



By the principle of linear superposition, the total electrostatic potential at point P is the sum of the potentials from each point charge:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where r_+ and r_- are the distances from the positive and negative charges to point P, respectively. Using the law of cosines:

$$r_+^2 = r^2 + \left(\frac{a}{2}\right)^2 - 2r\left(\frac{a}{2}\right)\cos\theta = r^2\left[1 - \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right]$$

$$r_-^2 = r^2 + \left(\frac{a}{2}\right)^2 - 2r\left(\frac{a}{2}\right)\cos(\pi - \theta) = r^2\left[1 + \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right]$$

Taking the reciprocal square root to find $1/r_+$ and $1/r_-$, and dropping the higher-order $a^2/4r^2$ term since $r \gg a$:

$$\frac{1}{r_+} = \frac{1}{r}\left[1 - \frac{a}{r}\cos\theta\right]^{-1/2}, \quad \frac{1}{r_-} = \frac{1}{r}\left[1 + \frac{a}{r}\cos\theta\right]^{-1/2}$$

Applying the standard binomial expansion formula $(1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \mathcal{O}(x^2)$ for $x \ll 1$:

$$\frac{1}{r_+} \approx \frac{1}{r}\left(1 + \frac{a}{2r}\cos\theta\right)$$
$$\frac{1}{r_-} \approx \frac{1}{r}\left(1 - \frac{a}{2r}\cos\theta\right)$$

Substituting these expansions back into the superposition potential equation:

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\left(1 + \frac{a}{2r}\cos\theta\right) - \left(1 - \frac{a}{2r}\cos\theta\right)\right]$$

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{a}{r}\cos\theta\right] = \frac{qa\cos\theta}{4\pi\epsilon_0 r^2}$$

By vector definition, the dot product between the dipole vector $\boldsymbol{p} = qa\hat{\boldsymbol{k}}$ and the position unit vector $\hat{\boldsymbol{r}}$ is $\boldsymbol{p}\cdot\boldsymbol{r} = pr\cos\theta = qar\cos\theta$. Therefore, we can rewrite the potential expression as:

$$V(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3}$$

This matches the requested functional template. By comparing terms, we find the required pre-factor constant α:

$$\alpha = \frac{1}{4\pi\epsilon_0}$$

Physics Hint: *Notice that while the potential of a single isolated point charge drops off as $1/r$, a dipole potential decays faster, scaling as $1/r^2$. This faster drop-off occurs because the equal and opposite point charges appear closer together when viewed from a far distance, causing their individual electric fields to partially cancel out.*

(b) Derivation of the Electric Field Vector in Spherical Polar Coordinates The electric field vector $\boldsymbol{E}$ is related to the scalar potential by the negative gradient operator ($\boldsymbol{E} = -\boldsymbol{\nabla} V$). In spherical polar coordinates (r, θ, ϕ), the gradient operator is defined as:

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Let us evaluate each component derivative independently using our derived potential function $V(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$:

- **Radial Component (E_r):**

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{4\pi\epsilon_0} r^{-2}\right) = -\left(\frac{p\cos\theta}{4\pi\epsilon_0}\cdot(-2)r^{-3}\right) = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

- **Polar Component (E_θ):**

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right) = -\frac{1}{4\pi\epsilon_0 r^3}\left(p\cdot(-\sin\theta)\right) = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

- **Azimuthal Component (E_ϕ):** Since the potential is perfectly symmetric about the z-axis, it has no functional dependence on the azimuthal angle ϕ:

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

Assembling these components gives the full electric field vector in spherical coordinates:

$$\boldsymbol{E}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

To express this electric field vector in a coordinate-independent form, we map the spherical unit vectors back to standard position vectors using the transformations $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ and $\hat{\boldsymbol{k}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \implies \sin\theta\,\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{r}} - \hat{\boldsymbol{k}}$:

$$2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}} = 2\cos\theta\,\hat{\boldsymbol{r}} + (\cos\theta\,\hat{\boldsymbol{r}} - \hat{\boldsymbol{k}}) = 3\cos\theta\,\hat{\boldsymbol{r}} - \hat{\boldsymbol{k}}$$

Multiplying through by the scalar dipole magnitude p gives:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^3}\left[3(p\cos\theta)\hat{\boldsymbol{r}} - p\hat{\boldsymbol{k}}\right] = \frac{1}{4\pi\epsilon_0 r^3}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2} - \boldsymbol{p}\right]$$

Physics Hint: *This coordinate-independent form is highly useful because it allows you to evaluate the electric field vector anywhere in space for any arbitrary orientation of the dipole vector* $\boldsymbol{p}$*, without being locked into a specific spherical polar coordinate alignment.*

Evaluation of the Spherical Vector Components As derived previously via the negative gradient of the electrostatic dipole potential ($V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$), the individual orthogonal components of the electric field vector in a spherical coordinate system are:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$
$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

This gives the full vector expression:

$$\boldsymbol{E} = E_r\hat{\boldsymbol{r}} + E_\theta\hat{\boldsymbol{\theta}} + E_\phi\hat{\boldsymbol{\phi}} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Derivation of the Absolute Scalar Magnitude $|\boldsymbol{E}|$ Because the basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ form a strictly orthonormal set ($\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}} = \delta_{ij}$), there are no cross-terms when squaring the vector. The absolute scalar magnitude $E = |\boldsymbol{E}|$ is computed by taking the square root of the sum of the squares of its individual components:

$$|\boldsymbol{E}| = \sqrt{E_r^2 + E_\theta^2 + E_\phi^2}$$

Substituting our coordinate functions into the norm equation:

$$|\boldsymbol{E}| = \sqrt{\left(\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}\right)^2 + \left(\frac{p\sin\theta}{4\pi\epsilon_0 r^3}\right)^2 + 0^2}$$

We factor out the shared scalar term $\frac{p}{4\pi\epsilon_0 r^3}$ from inside the radical:

$$|\boldsymbol{E}| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{(2\cos\theta)^2 + (\sin\theta)^2}$$

Expand the trigonometric terms inside the square root:

$$|\boldsymbol{E}| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

To simplify this into its standard textbook form, we use the fundamental Pythagorean trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$:

$$|\boldsymbol{E}| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{4\cos^2\theta + (1 - \cos^2\theta)}$$

Combining the remaining $\cos^2\theta$ terms yields the final expression for the absolute scalar magnitude:

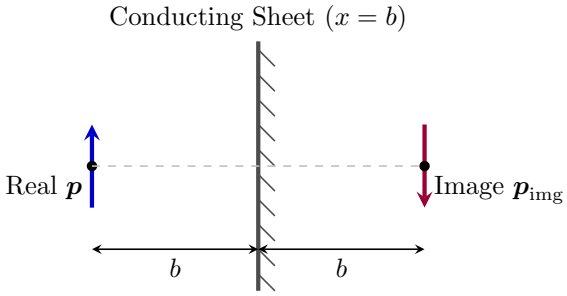
$$|\boldsymbol{E}| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\cos^2\theta + 1}$$

Physics Hint: *This scalar magnitude relation contains valuable physical intuition about field variations over an angular sweep. At any fixed radial shell distance r, the electric field reaches its maximum absolute strength along the dipole axis ($\theta = 0$ or π), where $\sqrt{3(1)+1} = 2$. Conversely, it drops to its minimum absolute strength along the equator ($\theta = \pi/2$), where $\sqrt{3(0)+1} = 1$. The axial field is exactly twice as strong as the equatorial field at the same distance.*

(c) Interaction with a Distant Infinite Conducting Sheet ($b \gg a$) An infinite grounded conducting sheet is placed in the plane $x = b$. To solve for the resulting electric field and potential distribution in the region $x < b$, we apply the **method of images**.

The boundary condition requires the surface of the conducting sheet to be an equipotential plane ($V = 0$). To satisfy this condition, we can replace the conducting sheet with an imaginary "mirror image" dipole $\boldsymbol{p}_{\text{img}}$ positioned symmetrically on the opposite side of the boundary at $x = 2b$.

Method of Images for a Parallel Dipole Layout



Let us determine the orientation of the image dipole. The real dipole consists of a $+q$ charge at $(0, 0, a/2)$ and a $-q$ charge at $(0, 0, -a/2)$. Reflecting these across the plane $x = b$ places their images at:

- Image of $+q$: carries a charge $-q$, located at $(2b, 0, a/2)$
- Image of $-q$: carries a charge $+q$, located at $(2b, 0, -a/2)$

The image dipole vector points from its negative charge (at $+z$) to its positive charge (at $-z$), yielding:

$$\boldsymbol{p}_{\text{img}} = -qa\hat{\boldsymbol{k}} = -\boldsymbol{p}$$

We now calculate the force and torque exerted on the real dipole by computing the electric field $\boldsymbol{E}_{\text{img}}$ generated by this image dipole at the origin $(0, 0, 0)$. The displacement vector from the image dipole to the real dipole is $\boldsymbol{R} = -2b\hat{\boldsymbol{i}}$, so $R = 2b$ and $\hat{\boldsymbol{R}} = -\hat{\boldsymbol{i}}$. Since $\boldsymbol{p}_{\text{img}} = -p\hat{\boldsymbol{k}}$, the dot product is $\boldsymbol{p}_{\text{img}} \cdot \hat{\boldsymbol{R}} = 0$. Substituting these values into the coordinate-independent field equation:

$$\boldsymbol{E}_{\text{img}}(0) = \frac{1}{4\pi\epsilon_0 R^3}\left[3(\boldsymbol{p}_{\text{img}} \cdot \hat{\boldsymbol{R}})\hat{\boldsymbol{R}} - \boldsymbol{p}_{\text{img}}\right] = \frac{1}{4\pi\epsilon_0(2b)^3}\left[\boldsymbol{0} - (-p\hat{\boldsymbol{k}})\right] = \frac{\boldsymbol{p}}{32\pi\epsilon_0 b^3} = \frac{qa}{32\pi\epsilon_0 b^3}\hat{\boldsymbol{k}}$$

(i) Force on the Dipole ($\boldsymbol{F}$): The net force exerted on a dipole in a non-uniform electric field depends on the spatial gradient of that field ($\boldsymbol{F} = (\boldsymbol{p} \cdot \boldsymbol{\nabla})\boldsymbol{E}$). Since $\boldsymbol{p} = p\hat{\boldsymbol{k}}$, this operator simplifies to $p\frac{\partial}{\partial z}$:

$$\boldsymbol{F} = p\frac{\partial \boldsymbol{E}_{\text{img}}}{\partial z}\bigg|_{(0,0,0)}$$

Because the image dipole sits a far distance $2b$ away along the x-axis, its field components near the origin vary with position. Let us evaluate the field at a small offset position $(0, 0, z)$ along the z-axis. The displacement vector becomes $\boldsymbol{R}' = -2b\hat{\boldsymbol{i}} + z\hat{\boldsymbol{k}}$, and its magnitude is $R' = (4b^2 + z^2)^{1/2}$. The dot product is $\boldsymbol{p}_{\text{img}} \cdot \boldsymbol{R}' = (-p\hat{\boldsymbol{k}}) \cdot (-2b\hat{\boldsymbol{i}} + z\hat{\boldsymbol{k}}) = -pz$. Substituting these into the field equation:

$$\boldsymbol{E}_{\text{img}}(z) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(-pz)(-2b\hat{\boldsymbol{i}} + z\hat{\boldsymbol{k}})}{(4b^2 + z^2)^{5/2}} - \frac{-p\hat{\boldsymbol{k}}}{(4b^2 + z^2)^{3/2}}\right]$$

Differentiating this expression with respect to z and evaluating at $z = 0$:

$$\left.\frac{\partial \boldsymbol{E}_{\text{img}}}{\partial z}\right|_{z=0} = \frac{1}{4\pi\epsilon_0}\left[\frac{6pb\hat{\boldsymbol{i}} - 0}{(4b^2)^{5/2}} - 0\right] = \frac{6pb\hat{\boldsymbol{i}}}{4\pi\epsilon_0(32b^5)} = \frac{3p}{64\pi\epsilon_0 b^4}\hat{\boldsymbol{i}}$$

Multiplying by the real dipole magnitude p gives the net attractive force:

$$\boldsymbol{F} = \frac{3p^2}{64\pi\epsilon_0 b^4}\hat{\boldsymbol{i}} = \frac{3q^2a^2}{64\pi\epsilon_0 b^4}\hat{\boldsymbol{i}}$$

(ii) Torque on the Dipole ($\boldsymbol{\tau}$): The torque experienced by a dipole in an external electric field is given by the vector cross product $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$. Substituting our values:

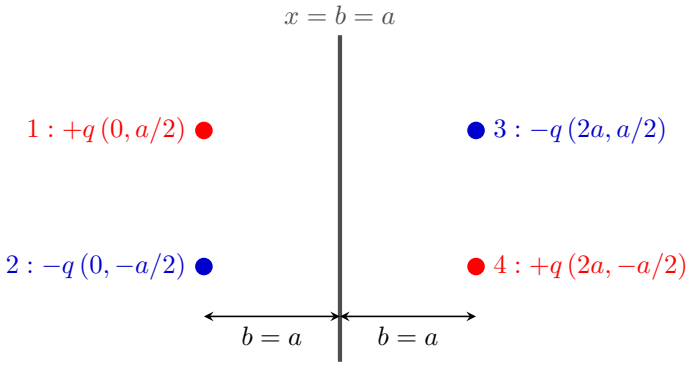
$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}_{\text{img}}(0) = (p\hat{\boldsymbol{k}}) \times \left(\frac{p}{32\pi\epsilon_0 b^3}\hat{\boldsymbol{k}}\right) = \boldsymbol{0} \quad \text{(Zero Net Torque)}$$

Physics Hint: *The net torque is exactly zero because the electric field generated by the image dipole at the origin points in the $+z$ direction, which is perfectly parallel to the real dipole vector. Since the vectors are collinear, their cross product vanishes.*

(d) Regime Shift for a Close Conducting Sheet ($b = a$) If the infinite conducting sheet is brought much closer to the dipole so that $b = a$, our previous far-field approximation ($b \gg a$) breaks down. We can no longer treat the system as two interacting point dipoles using ideal field equations, because the separation distance between the real charges and their images is now comparable to the internal size of the dipole itself.

To find the exact force, we must compute the individual Coulomb forces acting between all four discrete point charges in the system (the two real charges and their two image images).

Discrete Point Charge Interactions for Close Boundary



Let us evaluate the net force component along the x-axis acting on the real positive charge (Charge 1, located at $(0, a/2)$) due to the two image charges (Charge 3 at $(2a, a/2)$ and Charge 4 at $(2a, -a/2)$):

- **Force from Charge 3 ($-q$ at distance $2a$):** This is a direct attractive force along the $+x$ direction:

$$F_{13,x} = \frac{1}{4\pi\epsilon_0}\frac{q(-q)}{(2a)^2} \cdot (-1) = \frac{q^2}{16\pi\epsilon_0 a^2}$$

- **Force from Charge 4 ($+q$ at distance $\sqrt{(2a)^2 + a^2} = \sqrt{5}a$):** This is a repulsive force directed away from Charge 4. Its component along the x-axis is:

$$F_{14,x} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(\sqrt{5}a)^2}\cos\phi = -\frac{q^2}{20\pi\epsilon_0 a^2}\left(\frac{2a}{\sqrt{5}a}\right) = -\frac{2q^2}{20\sqrt{5}\pi\epsilon_0 a^2} = -\frac{q^2}{10\sqrt{5}\pi\epsilon_0 a^2}$$

By symmetry, the net force on the real negative charge (Charge 2) matches this profile exactly. Summing these discrete components gives the exact net attractive force acting on the entire dipole:

$$F_{\text{close}} = 2 \times (F_{13,x} + F_{14,x}) = \frac{2q^2}{4\pi\epsilon_0 a^2}\left[\frac{1}{4} - \frac{1}{5\sqrt{5}}\right] = \frac{q^2}{2\pi\epsilon_0 a^2}\left[\frac{1}{4} - \frac{1}{5\sqrt{5}}\right]$$

Torque Behavior: The net torque remains **exactly zero**. Even at this close distance, the entire point-charge geometry remains perfectly symmetric about the central x-axis line, meaning all turning forces cancel out out-of-phase.

Physics Hint: *This case demonstrates the limits of the ideal dipole approximation. When the boundary distance is comparable to the dipole's internal size, you must treat the system as a collection of discrete point charges rather than a single idealized point dipole to capture the exact physical behavior.*

Section B, Question 11

Problem Statement

State Maxwell's equations in their differential form assuming the presence of charge density ρ and current density $\boldsymbol{J}$, defining all terms that you use. [4]

Use one of these equations to derive Ampère's law and explain why a displacement current needs to be included. [3]

Show from Maxwell's equations that plane electromagnetic waves may propagate in a vacuum and derive their speed. [6]

A plane electromagnetic wave in a vacuum is described by an electric field given by

$$\boldsymbol{E} = (E_x, E_y, E_z) = (3E_0 \cos(kz - \omega t), 4E_0 \cos(kz - \omega t), 0)$$

where E_0, ω, and k are real positive constants. Determine the Cartesian components of $\boldsymbol{B}$ and calculate the numerical value of the ratio $\mu_0|\boldsymbol{E}|/|\boldsymbol{B}|$. [7]

Solution

(a) Maxwell's Equations in Differential Form In the presence of a local macroscopic free charge density ρ and a free current density vector field $\boldsymbol{J}$, the four fundamental differential Maxwell equations governing classical electrodynamics are written as:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's Law for Electricity)} \tag{4}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \text{(Gauss's Law for Magnetism)} \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{(Faraday's Law of Induction)} \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(Ampère-Maxwell Law)} \tag{7}$$

Where the physical vector and scalar quantities are defined as:

- $\boldsymbol{E}$: Electric field vector ($\mathrm{V} \cdot \mathrm{m}^{-1}$)
- $\boldsymbol{B}$: Magnetic flux density vector (T)
- ρ: Volumetric charge density ($\mathrm{C} \cdot \mathrm{m}^{-3}$)
- $\boldsymbol{J}$: Volumetric current density vector ($\mathrm{A} \cdot \mathrm{m}^{-2}$)
- ϵ_0: Permittivity of free space ($\approx 8.854 \times 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1}$)
- μ_0: Permeability of free space ($= 4\pi \times 10^{-7}\ \mathrm{H} \cdot \mathrm{m}^{-1}$)
- $\boldsymbol{\nabla}\cdot$: Divergence spatial operator (m^{-1})

- $\boldsymbol{\nabla}\times$: Curl spatial operator (m^{-1})

(b) Derivation of Ampère's Law and Necessity of Displacement Current To examine steady-state boundary conditions, consider the Ampère-Maxwell relation (Equation 4). In magnetostatic conditions, fields are invariant over time ($\frac{\partial \boldsymbol{E}}{\partial t} = \mathbf{0}$). This simplifies the relation to its classical static form:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

Applying Stokes' theorem integrates this local differential relation over an open surface S bounded by a closed contour perimeter curve C:

$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{A} = \oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$$

Equating this to the integrated source current density field yields the integral formulation of Ampère's Law:

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \iint_S \boldsymbol{J} \cdot d\boldsymbol{A} = \mu_0 I_{\text{enclosed}}$$

The Logical Necessity of the Displacement Current: Maxwell identified a profound mathematical contradiction in this static law when dealing with time-varying fields, such as charging a capacitor. Take the vector identity stating that the divergence of the curl of any arbitrary vector field is always identically zero:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = 0$$

If we apply this identity to the original static Ampère law ($\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$), we get:

$$\boldsymbol{\nabla} \cdot (\mu_0 \boldsymbol{J}) = 0 \implies \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

However, the localized conservation of electric charge is strictly governed by the continuity equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$

Thus, whenever charge density changes over time ($\frac{\partial \rho}{\partial t} \neq 0$), such as charge accumulating on a capacitor plate, $\boldsymbol{\nabla} \cdot \boldsymbol{J} \neq 0$. This creates an inconsistency, as the left side of the equation equals zero while the right side does not.

To resolve this contradiction, Maxwell substituted Gauss's Law ($\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$) into the continuity equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial}{\partial t}(\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}) = -\boldsymbol{\nabla} \cdot \left(\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

Rearranging terms under a single divergence operator yields:

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) = 0$$

This mathematically demonstrates that the combined quantity inside the parentheses is divergence-free under all physical conditions. Maxwell added this second term to Ampère's law as a new current equivalent, defining the **displacement current density** as $\boldsymbol{J}_d = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$. This term accounts for magnetic fields generated by changing electric fields, even in empty vacuum space where no physical charge carriers are present.

Physics Hint: *The displacement current is easily understood by looking at a charging capacitor. Between the plates, the conduction current I drops to zero because of the insulating gap. However, the electric field between the plates is changing rapidly. The displacement current density $\epsilon_0 \frac{\partial E}{\partial t}$ in this gap matches the external conduction current exactly, ensuring that the total current remains continuous across the circuit loop.*

(c) Derivation of the Electromagnetic Wave Equation in a Vacuum
In a perfect vacuum, there are no free electric charges ($\rho = 0$) and no free conduction currents ($\boldsymbol{J} = \boldsymbol{0}$). Under these conditions, Maxwell's equations simplify to:

$$\nabla \cdot \boldsymbol{E} = 0 \quad \text{(1a)} \qquad \nabla \cdot \boldsymbol{B} = 0 \quad \text{(2a)}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{(3a)} \qquad \nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(4a)}$$

To uncouple these vector equations, we apply the curl operator to Faraday's Law (Equation 3a):

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t} \right)$$

We can swap the order of the spatial and temporal derivatives on the right side. On the left side, we apply the standard vector identity $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$:

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B})$$

Substituting the vacuum source-free conditions (Equation 1a, $\nabla \cdot \boldsymbol{E} = 0$) and the Ampère-Maxwell relation (Equation 4a) into this expression:

$$\boldsymbol{0} - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

$$\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

An identical mathematical derivation can be performed for the magnetic field by taking the curl of Equation 4a, yielding:

$$\nabla^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

Both relations match the classical partial differential equation that describes wave propagation:

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

This demonstrates that time-varying electric and magnetic fields can propagate through a vacuum as self-sustaining electromagnetic waves. By comparing coefficients, we determine the propagation speed v:

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \implies v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

(d) Resolving the Cartesian Components of $\boldsymbol{B}$ We are given a plane electromagnetic wave propagating through a vacuum with its electric field vector defined as:

$$\boldsymbol{E}(z,t) = 3E_0\cos(kz-\omega t)\hat{\boldsymbol{i}} + 4E_0\cos(kz-\omega t)\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}$$

To determine the corresponding magnetic flux density vector $\boldsymbol{B}$, we use Faraday's Law ($\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$). Let us evaluate the curl of the given electric field vector:

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \det\begin{pmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{pmatrix} = \left(-\frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial E_x}{\partial z}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{k}}$$

Since the components vary only with the z coordinate, the partial derivatives with respect to x and y are zero. Let us compute the partial derivatives along z:

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial}{\partial z}\Big[4E_0\cos(kz-\omega t)\Big] = -4E_0\big(-\sin(kz-\omega t)\big)\cdot(k) = 4kE_0\sin(kz-\omega t)$$
$$\frac{\partial E_x}{\partial z} = \frac{\partial}{\partial z}\Big[3E_0\cos(kz-\omega t)\Big] = 3E_0\big(-\sin(kz-\omega t)\big)\cdot(k) = -3kE_0\sin(kz-\omega t)$$

This yields the curl of the electric field:

$$\boldsymbol{\nabla}\times\boldsymbol{E} = 4kE_0\sin(kz-\omega t)\hat{\boldsymbol{i}} - 3kE_0\sin(kz-\omega t)\hat{\boldsymbol{j}}$$

Now, equate this to the time derivative of the magnetic field according to Faraday's law:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\left(\boldsymbol{\nabla}\times\boldsymbol{E}\right) = -4kE_0\sin(kz-\omega t)\hat{\boldsymbol{i}} + 3kE_0\sin(kz-\omega t)\hat{\boldsymbol{j}}$$

To isolate $\boldsymbol{B} = (B_x, B_y, B_z)$, we integrate each component function independently with respect to time t:

$$B_x = \int -4kE_0\sin(kz-\omega t)\,dt = -4kE_0\left[\frac{-\cos(kz-\omega t)}{-\omega}\right] = -\frac{4kE_0}{\omega}\cos(kz-\omega t)$$
$$B_y = \int 3kE_0\sin(kz-\omega t)\,dt = 3kE_0\left[\frac{-\cos(kz-\omega t)}{-\omega}\right] = \frac{3kE_0}{\omega}\cos(kz-\omega t)$$
$$B_z = 0$$

Using the wave propagation relation $\frac{k}{\omega} = \frac{1}{c}$, the components of the magnetic field simplify to:

$$B_x = -\frac{4E_0}{c}\cos(kz-\omega t)$$
$$B_y = \frac{3E_0}{c}\cos(kz-\omega t)$$
$$B_z = 0$$

(e) Calculation of the Ratio $\mu_0|\boldsymbol{E}|/|\boldsymbol{B}|$ Let us find the absolute scalar magnitudes of both vector fields:

- **Electric Field Magnitude ($|\boldsymbol{E}|$):**

$$|\boldsymbol{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2} = \sqrt{[3E_0\cos(kz-\omega t)]^2 + [4E_0\cos(kz-\omega t)]^2 + 0}$$

$$|\boldsymbol{E}| = \sqrt{(9+16)E_0^2\cos^2(kz-\omega t)} = \sqrt{25E_0^2\cos^2(kz-\omega t)} = 5E_0|\cos(kz-\omega t)|$$

- **Magnetic Field Magnitude ($|\boldsymbol{B}|$):**

$$|\boldsymbol{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{\left[-\frac{4E_0}{c}\cos(kz-\omega t)\right]^2 + \left[\frac{3E_0}{c}\cos(kz-\omega t)\right]^2 + 0}$$

$$|\boldsymbol{B}| = \sqrt{(16+9)\frac{E_0^2}{c^2}\cos^2(kz-\omega t)} = \sqrt{25\frac{E_0^2}{c^2}\cos^2(kz-\omega t)} = \frac{5E_0}{c}|\cos(kz-\omega t)|$$

Now, calculate the ratio of the field magnitudes:

$$\frac{|\boldsymbol{E}|}{|\boldsymbol{B}|} = \frac{5E_0|\cos(kz-\omega t)|}{\frac{5E_0}{c}|\cos(kz-\omega t)|} = c$$

Using this result, we determine the final target ratio $\frac{\mu_0|\boldsymbol{E}|}{|\boldsymbol{B}|}$:

$$\frac{\mu_0|\boldsymbol{E}|}{|\boldsymbol{B}|} = \mu_0 c$$

To calculate the exact numerical value of this ratio, we substitute the definition of the speed of light ($c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$):

$$\mu_0 c = \mu_0\left(\frac{1}{\sqrt{\mu_0\epsilon_0}}\right) = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0$$

where Z_0 is the characteristic impedance of free space. Substituting the numerical values of the constants:

$$Z_0 = \sqrt{\frac{4\pi\times10^{-7}\ \mathrm{H\cdot m^{-1}}}{8.8541878\times10^{-12}\ \mathrm{F\cdot m^{-1}}}} \approx 376.73\,\Omega$$

Rounding properly to three significant figures yields the final value:

$$\frac{\mu_0|\boldsymbol{E}|}{|\boldsymbol{B}|} \approx 377\,\Omega$$

Physics Hint: *The ratio $|\boldsymbol{E}|/|\boldsymbol{B}| = c$ is a fundamental property of plane electromagnetic waves propagating through a vacuum. When multiplied by μ_0, this ratio equals the characteristic impedance of free space ($Z_0 \approx 377\,\Omega$). This ratio of the transverse electric field to the transverse magnetic field of an electromagnetic wave in empty space is a constant..*