CP1 2024 — Study Guide

Question 1

Problem Statement

1. (a) Derive the moment of inertia of a thin solid disc, of uniform density, about an axis through its centre of mass and perpendicular to its face (the 'principal axis'); assume the disc mass is M and its radius is R. **[3]**

(b) State the parallel axis and perpendicular axis theorems. Hence determine the moment of inertia of the disc about an axis perpendicular to its principal axis and tangential to the rim of the disc (assume the disc is negligibly thin). **[4]**

Complete Solution & Study Guide

(a) Derivation of the Principal Moment of Inertia

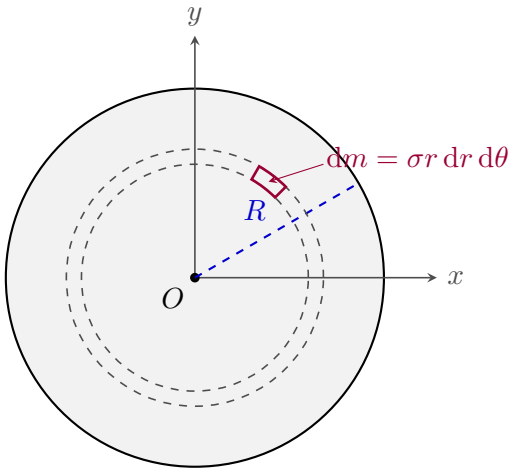
Let a thin solid disc have a uniform surface mass density σ. The total mass M of the disc is distributed over its flat surface area $A = \pi R^2$. Therefore, we express the uniform surface density as:

$$\sigma = \frac{M}{\pi R^2} \tag{1}$$

An infinitesimal area element $\mathrm{d}A$ in polar coordinates is given by the differential area $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. The corresponding differential mass element $\mathrm{d}m$ is:

$$\mathrm{d}m = \sigma\,\mathrm{d}A = \sigma r\,\mathrm{d}r\,\mathrm{d}\theta \tag{2}$$

Disc Coordinate System and Mass Elements



Evaluating the principal moment of inertia integral:

$$I_z = \int_0^{2\pi} \int_0^R r^2 \left(\sigma r \, \mathrm{d}r \, \mathrm{d}\theta\right) = \sigma \left(\int_0^{2\pi} \mathrm{d}\theta\right) \left(\int_0^R r^3 \, \mathrm{d}r\right) \tag{3}$$

$$I_z = \sigma \cdot (2\pi) \cdot \left(\frac{R^4}{4}\right) = \frac{\pi \sigma R^4}{2} \tag{4}$$

Substituting $\sigma = \frac{M}{\pi R^2}$ yields:

$$I_{\mathrm{CM},z} = \frac{1}{2} M R^2 \tag{5}$$

(b) Theorem Formulations and Off-Axis Calculations

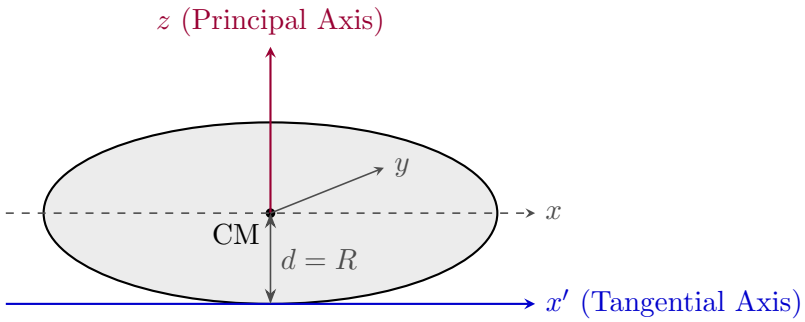
1. **The Parallel Axis Theorem**: $I = I_{\mathrm{CM}} + Md^2$
2. **The Perpendicular Axis Theorem**: $I_z = I_x + I_y$

By planar symmetry for a uniform circular disk, $I_{\mathrm{CM},x} = I_{\mathrm{CM},y}$. From the perpendicular axis theorem:

$$I_{\mathrm{CM},z} = 2I_{\mathrm{CM},x} \implies \frac{1}{2} M R^2 = 2I_{\mathrm{CM},x} \implies I_{\mathrm{CM},x} = \frac{1}{4} M R^2 \tag{6}$$

To find the moment of inertia about an axis tangential to the rim (x') and parallel to the in-plane diameter axis (x), the translation distance is $d = R$.

Disc Rotational Axes Configuration



Applying the parallel axis theorem:

$$I_{\mathrm{tangent}} = I_{\mathrm{CM},x} + Md^2 = \frac{1}{4} M R^2 + M(R)^2 = \frac{5}{4} M R^2 \tag{7}$$

Question 2

Problem Statement

2. A thin uniform rod of mass M and length L lies on a frictionless horizontal table. A particle, also of mass M, moves along the table top at a constant (non-relativistic) speed u and strikes the rod perpendicular to its axis at a certain distance y from its centre.

Calculate this distance such that the particle and the centre of mass of the rod move with the same velocity after the collision. The collision can be assumed to be perfectly elastic; the moment of inertia of a thin rod about an axis through its centre of mass and perpendicular to its length is $\frac{1}{12}ML^2$. **[6]**

Complete Solution & Study Guide

Physical Setup & General Principles

We analyze this problem using three fundamental conservation laws of classical mechanics for an isolated system:

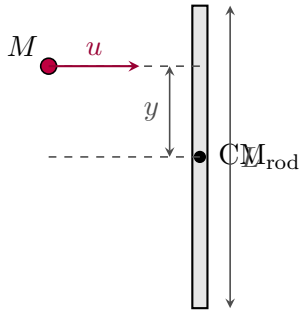
1. **Conservation of Linear Momentum**: The horizontal table surface is completely frictionless, meaning no external net forces act on the rod-particle system.

2. **Conservation of Angular Momentum**: Because there are no external forces, there are no external torques acting on the system. We choose to evaluate angular momentum about the fixed position corresponding to the center of mass (CM) of the rod.

3. **Conservation of Kinetic Energy**: The problem explicitly states that the collision is *perfectly elastic*, meaning mechanical kinetic energy is conserved before and after impact.

Physics Hint: *Choosing the target's initial center of mass as the origin for angular momentum simplifies the math because the rod's initial linear momentum contributes zero torque about its own center, and its initial angular momentum is likewise zero.*

Rod-Particle Impact Geometry



Step-by-Step Algebraic Derivation

Step 1: Application of Conservation of Linear Momentum Let the mass of the particle be M and the mass of the rod be M. Before the collision, the particle travels with a constant initial velocity u and the rod is entirely at rest ($v_{\text{rod},i} = 0$). After the collision, let the final velocity of the particle be v_p and the final linear velocity of the center of mass of the rod be v_{CM}.

The constraint given in the problem statement requires that the particle and the center of mass of the rod move with the *same velocity* after the collision:

$$v_p = v_{\text{CM}} = v \tag{8}$$

Setting up the statement for the conservation of linear momentum along the axis of the particle's initial trajectory:

$$P_i = P_f \implies Mu = Mv_p + Mv_{\text{CM}} \tag{9}$$

Substitute the structural condition $v_p = v_{\text{CM}} = v$:

$$Mu = Mv + Mv = 2Mv \tag{10}$$

Since $M \neq 0$, we divide both sides by the mass M to isolate the post-collision velocity parameter v:

$$u = 2v \implies v = \frac{u}{2} \tag{11}$$

Step 2: Application of Conservation of Kinetic Energy Because the collision is perfectly elastic, the total initial mechanical kinetic energy equals the total final mechanical kinetic energy. The final kinetic energy of the system consists of three parts: the linear translational kinetic energy of the particle, the linear translational kinetic energy of the rod's center of mass, and the rotational kinetic energy of the rod about its own center of mass.

$$K_i = K_f \implies \frac{1}{2}Mu^2 = \frac{1}{2}Mv_p^2 + \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2 \tag{12}$$

Substitute the known post-collision velocity values $v_p = v_{\text{CM}} = v = \frac{u}{2}$ into the kinetic energy equation:

$$\frac{1}{2}Mu^2 = \frac{1}{2}M\left(\frac{u}{2}\right)^2 + \frac{1}{2}M\left(\frac{u}{2}\right)^2 + \frac{1}{2}I_{\text{CM}}\omega^2 \tag{13}$$

Simplify the translational terms on the right-hand side:

$$\frac{1}{2}Mu^2 = 2 \times \left[\frac{1}{2}M\left(\frac{u^2}{4}\right)\right] + \frac{1}{2}I_{\text{CM}}\omega^2 \tag{14}$$

$$\frac{1}{2}Mu^2 = \frac{1}{4}Mu^2 + \frac{1}{2}I_{\text{CM}}\omega^2 \tag{15}$$

Subtract $\frac{1}{4}Mu^2$ from both sides of the equation to isolate the rotational kinetic energy term:

$$\left(\frac{1}{2} - \frac{1}{4}\right)Mu^2 = \frac{1}{2}I_{\text{CM}}\omega^2 \implies \frac{1}{4}Mu^2 = \frac{1}{2}I_{\text{CM}}\omega^2 \tag{16}$$

We are given that the moment of inertia of the thin uniform rod about its perpendicular center of mass axis is $I_{\text{CM}} = \frac{1}{12}ML^2$. Substituting this parameter expression:

$$\frac{1}{4}Mu^2 = \frac{1}{2}\left(\frac{1}{12}ML^2\right)\omega^2 \tag{17}$$

$$\frac{1}{4}Mu^2 = \frac{1}{24}ML^2\omega^2 \tag{18}$$

Cancel out the common mass factor M from both sides:

$$\frac{1}{4}u^2 = \frac{1}{24}L^2\omega^2 \tag{19}$$

Isolate ω^2 by multiplying both sides by 24:

$$u^2 = \frac{4}{24}L^2\omega^2 \implies u^2 = \frac{1}{6}L^2\omega^2 \tag{20}$$

$$\omega^2 = \frac{6u^2}{L^2} \implies \omega = \frac{\sqrt{6}u}{L} \tag{21}$$

Physics Hint: *The term ω represents the final angular velocity of the rod. Notice how it increases linearly with the initial impact speed u and varies inversely with the rod's scale length L, matching physical expectations.*

Step 3: Application of Conservation of Angular Momentum We calculate the angular momentum about the fixed point corresponding to the center of mass of the rod.

- **Before impact**: The rod is stationary. The particle moves with linear momentum Mu along a path whose perpendicular distance (impact parameter) to the rod's CM is y. Thus, the initial angular momentum is $L_i = Muy$.
- **After impact**: The particle moves with linear momentum $Mv_p = Mv$ at the same distance y, contributing an angular momentum of Mvy. The rod translates with momentum Mv_{CM} (which contributes zero angular momentum about its own center of mass) and rotates with angular velocity ω, contributing a spin angular momentum of $I_{\text{CM}}\omega$.

Equating initial and final angular momentum:

$$L_i = L_f \implies Muy = Mvy + I_{\text{CM}}\omega \tag{22}$$

Substitute the known linear velocity $v = \frac{u}{2}$ and the rod's moment of inertia $I_{\text{CM}} = \frac{1}{12}ML^2$:

$$Muy = M\left(\frac{u}{2}\right)y + \left(\frac{1}{12}ML^2\right)\omega \tag{23}$$

Subtract $\frac{1}{2}Muy$ from both sides to gather the particle terms:

$$\left(1 - \frac{1}{2}\right)Muy = \frac{1}{12}ML^2\omega \implies \frac{1}{2}Muy = \frac{1}{12}ML^2\omega \tag{24}$$

Cancel out the common factor of mass M:

$$\frac{1}{2}uy = \frac{1}{12}L^2\omega \tag{25}$$

Now, substitute the expression for ω that we derived from energy conservation ($\omega = \frac{\sqrt{6}u}{L}$):

$$\frac{1}{2}uy = \frac{1}{12}L^2\left(\frac{\sqrt{6}u}{L}\right) \tag{26}$$

Simplify the right-hand side by cancelling out one factor of L:

$$\frac{1}{2}uy = \frac{\sqrt{6}}{12}Lu \tag{27}$$

Since the initial speed $u \neq 0$, we can cancel u from both sides of the equation:

$$\frac{1}{2}y = \frac{\sqrt{6}}{12}L \tag{28}$$

Isolate the impact distance parameter y by multiplying both sides by 2:

$$y = 2 \times \frac{\sqrt{6}}{12}L = \frac{\sqrt{6}}{6}L \tag{29}$$

We rationalize the fraction to its standard, simplified radical form:

$$y = \frac{\sqrt{6}}{\sqrt{6} \times \sqrt{6}}L = \frac{1}{\sqrt{6}}L \tag{30}$$

Final Analytical Result

The specific distance from the center of the rod where the particle must strike to fulfill the post-collision velocity constraint is:

$$y = \frac{L}{\sqrt{6}} \approx 0.4082L \tag{31}$$

Physics Hint: *Since $y \approx 0.408L$ and the total half-length of the rod from its center to its physical tip is $0.5L$, this impact point lies inside the boundary of the rod. This shows the result is physically realistic and achievable.*

Question 3

Problem Statement

3. (a) An athlete throws a ball of mass M vertically upwards. The ball leaves her hand with speed u and experiences a force due to air resistance proportional to its speed, with proportionality constant K. Derive an expression for the time taken for the ball to reach its maximum height in terms of K. Assume the acceleration due to gravity is constant with height. **[5]**

(b) Now consider if the athlete were standing on another planet identical to Earth but with no atmosphere such that $K = 0$. By using your result from (a) and considering the limit $K \to 0$, or otherwise, calculate the time taken for the ball to reach its maximum height. **[2]**

Complete Solution & Study Guide

(a) Derivation of Time to Maximum Height with Linear Drag

Physical Setup and Equation of Motion We define a coordinate axis with the positive vertical direction pointing upward. Let y be the vertical position and $v = \frac{\mathrm{d}y}{\mathrm{d}t}$ be the instantaneous velocity of the ball. As the ball travels upward, two downward forces act upon it:

1. The constant force of gravity: $F_g = -Mg$.
2. The linear drag or air resistance force: $F_{\text{drag}} = -Kv$, which acts opposite to the velocity vector.

Physics Hint: *Because the ball is moving strictly upwards during this phase ($v > 0$), the drag force points downwards, reinforcing gravity. Thus, both terms appear with negative signs in Newton's second law.*

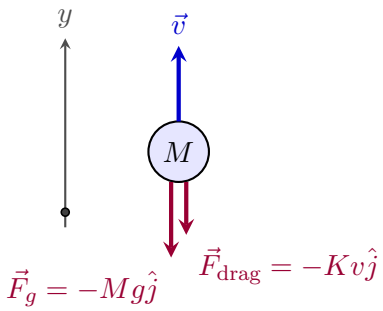
Using Newton's Second Law ($F_{\text{net}} = Ma = M\frac{\mathrm{d}v}{\mathrm{d}t}$):

$$-Mg - Kv = M\frac{\mathrm{d}v}{\mathrm{d}t} \tag{32}$$

To systematically solve this differential equation, we isolate the derivative term $\frac{\mathrm{d}v}{\mathrm{d}t}$ by dividing both sides by the total mass M:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{K}{M}v \tag{33}$$

Free Body Diagram of Upward Flight



Separation of Variables and Integral Setup We rearrange the differential equation to separate the kinematic variables, grouping all terms containing velocity v on the left-hand side and the differential time $\mathrm{d}t$ on the right-hand side:

$$\frac{\mathrm{d}v}{g + \frac{K}{M}v} = -\mathrm{d}t \tag{34}$$

We define our integration limits explicitly based on the physical boundaries of the upward flight path:

- At time $t = 0$, the ball leaves the hand with an initial speed $v = u$.
- At time $t = T$, the ball reaches its peak trajectory (maximum height), where its instantaneous vertical velocity becomes exactly $v = 0$.

Integrating both sides over these corresponding physical domains:

$$\int_u^0 \frac{\mathrm{d}v}{g + \frac{K}{M}v} = \int_0^T -\mathrm{d}t \tag{35}$$

Evaluating the right-hand side is straightforward:

$$\int_0^T -\mathrm{d}t = \Big[-t\Big]_0^T = -T \tag{36}$$

To easily compute the left-hand side integral, perform a substitution or recall the standard form $\int \frac{1}{a+bx}\,\mathrm{d}x = \frac{1}{b}\ln|a+bx|$:

$$\left[\frac{1}{\left(\frac{K}{M}\right)}\ln\left(g + \frac{K}{M}v\right)\right]_u^0 = -T \tag{37}$$

$$\frac{M}{K}\left[\ln(g) - \ln\left(g + \frac{K}{M}u\right)\right] = -T \tag{38}$$

Multiply both sides by -1 to swap the order of terms inside the bracket:

$$T = \frac{M}{K}\left[\ln\left(g + \frac{K}{M}u\right) - \ln(g)\right] \tag{39}$$

Using the standard logarithmic identity $\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$, we condense the expression:

$$T = \frac{M}{K}\ln\left(\frac{g + \frac{K}{M}u}{g}\right) \tag{40}$$

Distribute the denominator g across the terms inside the logarithm:

$$T = \frac{M}{K}\ln\left(1 + \frac{Ku}{Mg}\right) \tag{41}$$

Thus, the exact analytical expression for the time taken to reach maximum height under linear drag is:

$$T = \frac{M}{K}\ln\left(1 + \frac{Ku}{Mg}\right) \tag{42}$$

(b) Limit Analysis as Atmosphere Vanishes ($K \to 0$)

Method 1: Maclaurin Series Expansion (Limit Check) When evaluating the limit as $K \to 0$, a naive substitution yields an indeterminate form of the type $0 \times \infty$. To resolve this, we utilize the Maclaurin series expansion for the function $\ln(1 + x)$:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \mathcal{O}(x^4) \quad \text{for } |x| < 1 \tag{43}$$

In our timing expression, let the small parameter asset be $x = \frac{Ku}{Mg}$. As $K \to 0$, $x \to 0$. Substituting the first-order approximation into our timing expression:

$$T = \lim_{K\to 0} \frac{M}{K}\left[\left(\frac{Ku}{Mg}\right) - \frac{1}{2}\left(\frac{Ku}{Mg}\right)^2 + \dots\right] \tag{44}$$

Distribute the leading factor $\frac{M}{K}$ into the leading terms of the series:

$$T = \lim_{K\to 0}\left[\left(\frac{M}{K}\cdot\frac{Ku}{Mg}\right) - \frac{M}{2K}\cdot\frac{K^2u^2}{M^2g^2} + \dots\right] \tag{45}$$

$$T = \lim_{K\to 0}\left[\frac{u}{g} - \frac{Ku^2}{2Mg^2} + \dots\right] \tag{46}$$

Taking the limit explicitly as $K \to 0$, all terms containing residual factors of K in the numerator vanish cleanly:

$$T_0 = \frac{u}{g} \tag{47}$$

Method 2: Direct Solution via Kinematics (Otherwise) Alternatively, if there is no atmosphere, $K = 0$. The original differential equation simplifies directly to basic constant gravity:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g \implies \mathrm{d}v = -g\,\mathrm{d}t \tag{48}$$

Integrating directly using identical boundary limits ($v = u$ at $t = 0$, and $v = 0$ at $t = T_0$):

$$\int_u^0 \mathrm{d}v = -g \int_0^{T_0} \mathrm{d}t \implies 0 - u = -gT_0 \tag{49}$$

$$-u = -gT_0 \implies T_0 = \frac{u}{g} \tag{50}$$

Physics Hint: *The fact that the series expansion matches the elementary kinematic formula $v = u - gt$ validates the algebraic integrity of our logarithmic drag framework.*

Question 4

Problem Statement

4. (a) A system comprises a positively-charged particle of charge $+q_e e$, where e is the proton charge, and a nucleus of atomic number Z separated by a distance d. Write down the electrical potential energy of this system. **[1]**

(b) Now consider the particle to be accelerated from rest, starting at infinity, through a potential difference V and directed towards the nucleus, which is fixed in a target. Calculate the minimum distance of closest approach, stating any assumptions made. **[4]**

(c) Protons with kinetic energy 4.2 MeV are fired at a fixed gold target. Evaluate the minimum distance of closest approach of a proton to a gold nucleus; the atomic number of gold is 79. Compare this value with the typical size of an atom. **[2]**

(d) If α particles (which contain 2 protons) with the same kinetic energy, 4.2 MeV, were used instead of protons (as in Rutherford's famous scattering experiment), what would be the distance of closest approach? **[1]**

Complete Solution & Study Guide

(a) Electrical Potential Energy Formulation

By applying Coulomb's Law, the electrostatic potential energy U of two point charges separated by a distance d in a vacuum is given by:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{d} \tag{51}$$

Here, the moving incoming particle carries a net positive charge $q_1 = +q_e e$, and the stationary target nucleus carries a net positive charge proportional to its atomic number, $q_2 = +Ze$. Substituting these specific charges into the potential energy equation:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{(q_e e)(Ze)}{d} = \frac{q_e Z e^2}{4\pi\varepsilon_0 d} \tag{52}$$

Physics Hint: *Because both interacting bodies carry positive electrical charges, the resulting energy $U > 0$, confirming that the electrostatic interaction is purely repulsive. Work must be performed on the system to bring the two particles closer together.*

(b) Minimum Distance of Closest Approach: General Expression

Physical Assumptions To model this scattering interaction analytically, we make two standard structural assumptions:

1. **Head-on Collision**: The incoming particle is directed precisely along a radial line towards the center of the nucleus (impact parameter $b = 0$). This ensures the velocity vectors lie entirely on a one-dimensional line, meaning all initial kinetic energy can convert entirely into electrical potential energy at the turning point.

2. **Fixed Target / Infinite Mass Recoil**: The target nucleus is considered infinitely heavy compared to the bullet particle ($M_{\text{nucleus}} \gg m_{\text{particle}}$), allowing us to neglect target recoil and treat the nucleus as an immovable, static source of potential field.

Step-by-Step Algebraic Derivation The charged particle accelerates from rest at infinity through a potential difference V. The work done by the electric field equals the kinetic energy K_i gained by the particle:

$$K_i = q_{\text{particle}} V = q_e e V \tag{53}$$

As the particle travels towards the target nucleus, it experiences a repulsive force that slows it down. By Conservation of Mechanical Energy, the total mechanical energy must remain constant throughout the trajectory:

$$E_i = E_f \implies K_i + U_i = K_f + U_f \tag{54}$$

At infinity, the electrostatic potential energy is zero ($U_i = 0$). At the distance of closest approach ($d = d_{\text{min}}$), the particle stops instantaneously before reversing direction, so its final kinetic energy is zero ($K_f = 0$).

Equating the initial kinetic energy to the maximum potential energy at the turning point:

$$q_e e V = \frac{q_e Z e^2}{4\pi\varepsilon_0 d_{\text{min}}} \tag{55}$$

We isolate the minimum approach distance d_{min} by dividing both sides by the common charge factor $q_e e$:

$$V = \frac{Ze}{4\pi\varepsilon_0 d_{\text{min}}} \tag{56}$$

$$d_{\text{min}} = \frac{Ze}{4\pi\varepsilon_0 V} \tag{57}$$

(c) Evaluation for Protons hitting a Gold Target

Given Numerical Values

- Particle type: Proton $\implies q_e = 1$.
- Initial kinetic energy: $K_i = 4.2$ MeV $= 4.2 \times 10^6$ eV.
- Target nucleus: Gold (Au) $\implies Z = 79$.
- Fundamental Constant: $\frac{1}{4\pi\varepsilon_0} \approx 8.988 \times 10^9$ N $\cdot$ m$^2 \cdot$ C^{-2}.
- Elementary charge: $e \approx 1.602 \times 10^{-19}$ C.

Arithmetic Calculation Using the energy balance equation directly with kinetic energy explicitly stated in electron-volts (eV):

$$K_i = \frac{Ze^2}{4\pi\varepsilon_0 d_{\min}} \tag{58}$$

To match units efficiently, convert the energy from eV to Joules (J) by multiplying by e, or express the equation with one factor of e canceled:

$$(4.2 \times 10^6 \text{ eV}) \cdot (1.602 \times 10^{-19} \text{ J/eV}) = \frac{(8.988 \times 10^9) \cdot 79 \cdot (1.602 \times 10^{-19})^2}{d_{\min}} \tag{59}$$

Equivalently, canceling one factor of e from both sides simplifies the numerical execution:

$$4.2 \times 10^6 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Ze}{d_{\min}} \implies d_{\min} = \frac{(8.988 \times 10^9) \cdot 79 \cdot (1.602 \times 10^{-19})}{4.2 \times 10^6} \tag{60}$$

$$d_{\min} = \frac{1.1375 \times 10^{-7}}{4.2 \times 10^6} \approx 2.708 \times 10^{-14} \text{ m} \tag{61}$$

Thus, the minimum distance of closest approach for a proton is:

$$d_{\min,\text{ proton}} \approx 2.7 \times 10^{-14} \text{ m} \tag{62}$$

Comparison with Atomic Scale A typical atom has a spatial diameter on the order of an Angstrom:

$$R_{\text{atom}} \approx 10^{-10} \text{ m} \tag{63}$$

Comparing our calculated distance to the atomic scale:

$$\frac{d_{\min}}{R_{\text{atom}}} = \frac{2.7 \times 10^{-14} \text{ m}}{10^{-10} \text{ m}} = 2.7 \times 10^{-4} \tag{64}$$

The distance of closest approach is roughly 4 orders of magnitude smaller than the typical size of an atom ($\approx 10^{-10}$ m), demonstrating that the particle penetrates deep past the electron cloud, probing the subatomic core.

(d) Evaluation for Alpha Particles

An alpha particle (α) contains 2 protons, meaning its structural charge coefficient is $q_e = 2$. The problem states that the alpha particle has the *same initial kinetic energy* as the proton ($K_i = 4.2$ MeV). We write out the energy balance equation for this new scenario:

$$K_i = \frac{(2)Ze^2}{4\pi\varepsilon_0 d_{\text{min, }\alpha}} \tag{65}$$

Let us express this directly in terms of the proton distance of closest approach ($d_{\text{min, proton}}$) under identical kinetic energy constraints:

$$K_i = \frac{2Ze^2}{4\pi\varepsilon_0 d_{\text{min, }\alpha}} = \frac{Ze^2}{4\pi\varepsilon_0 d_{\text{min, proton}}} \tag{66}$$

Canceling out the common factors ($\frac{Ze^2}{4\pi\varepsilon_0}$):

$$\frac{2}{d_{\text{min, }\alpha}} = \frac{1}{d_{\text{min, proton}}} \implies d_{\text{min, }\alpha} = 2\, d_{\text{min, proton}} \tag{67}$$

Because the alpha particle carries double the electric charge of a proton, it experiences twice the repulsive force at any given distance, causing it to consume its kinetic energy faster and stop further out from the nucleus.

Numerical Output

$$d_{\text{min, }\alpha} = 2 \times (2.708 \times 10^{-14}\ \text{m}) \approx 5.42 \times 10^{-14}\ \text{m} \tag{68}$$

Question 5

Problem Statement

5. **(a)** Write down the Lorentz transformations for x and t between a reference frame S' moving with speed v along the x-axis with respect to a reference frame S. **[2]**

(b) In the moving frame S', two spatially-separated events occur simultaneously. What is the relationship between the relative times and relative locations of the two events in frame S? **[2]**

(c) What is the minimum possible spatial separation that these two events can have in frame S if their time separation in that frame is 1 nanosecond, and for what value of v is this the case? **[2]**

Complete Solution & Study Guide

(a) Standard Lorentz Transformations

Consider two inertial reference frames, S and S', in a standard configuration where frame S' translates with a constant velocity v along the positive x-axis direction relative to S. Assuming that the spacetime origins coincide at $t = t' = 0$, the coordinate mapping between (x, y, z, t) in S and (x', y', z', t') in S' is governed by the standard Lorentz transformations:

$$x' = \gamma(x - vt) \tag{69}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{70}$$

where γ is the dimensionless Lorentz factor defined by:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{71}$$

The transverse spatial components remain unaffected by the choice of relative motion axis: $y' = y$ and $z' = z$.

Physics Hint: *The structural symmetry of Special Relativity implies that the inverse transformations—mapping variables from S' back into S—can be written immediately by changing the sign of the relative velocity parameter ($v \to -v$) and swapping the primed and unprimed coordinates:* $x = \gamma(x' + vt')$ *and* $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$.

(b) Relativity of Simultaneity and Event Spacing

Step-by-Step Algebraic Derivation Let two distinct spacetime events be designated as Event 1 and Event 2. We define their relative coordinate intervals in any frame using the standard delta notation: $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$.

We are given that the two events occur *simultaneously* within the moving reference frame S'. This means their internal temporal separation interval is zero:

$$\Delta t' = t'_2 - t'_1 = 0 \tag{72}$$

Furthermore, they are specified to be *spatially separated* in S', meaning:

$$\Delta x' = x'_2 - x'_1 \neq 0 \tag{73}$$

To determine the relationship between their measured separation coordinates in the laboratory frame S, we apply the linear differential version of the inverse Lorentz transformation for time:

$$\Delta t = \gamma \left(\Delta t' + \frac{v \Delta x'}{c^2} \right) \tag{74}$$

Substituting the simultaneity condition $\Delta t' = 0$ yields:

$$\Delta t = \gamma \left(0 + \frac{v \Delta x'}{c^2} \right) = \frac{\gamma v \Delta x'}{c^2} \tag{75}$$

Next, we write the corresponding spatial transformation interval mapping from S' to S:

$$\Delta x = \gamma (\Delta x' + v \Delta t') \tag{76}$$

Substituting $\Delta t' = 0$ reduces this relationship to simple length contraction dilation scaling:

$$\Delta x = \gamma \Delta x' \implies \Delta x' = \frac{\Delta x}{\gamma} \tag{77}$$

We combine these two expressions to eliminate the intermediate frame S' track parameter $\Delta x'$. Substitute $\Delta x' = \frac{\Delta x}{\gamma}$ back into our temporal equation:

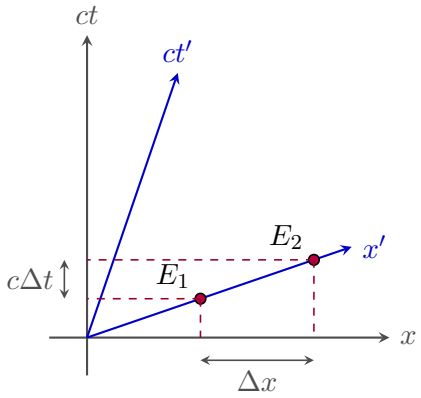
$$\Delta t = \frac{\gamma v}{c^2} \left(\frac{\Delta x}{\gamma} \right) \tag{78}$$

Cancelling out the common Lorentz scaling factor γ yields the final direct relationship between the intervals observed in frame S:

$$\Delta t = \frac{v \Delta x}{c^2} \tag{79}$$

Physics Hint: *This equation illustrates the core physical concept of the relativity of simultaneity. Two events that happen at the same time but at different places in a moving frame cannot appear simultaneous in the laboratory frame. Their time order shifts linearly with their spatial distribution.*

Spacetime Diagrams and Frame Projections



(c) Extremization of Spatial Separation

Step-by-Step Algebraic Minimization We are asked to minimize the spatial separation Δx in frame S given a fixed measured time separation of $\Delta t = 1$ nanosecond $= 10^{-9}$ s. Isolating the coordinate separation parameter Δx from the equation derived in part (b):

$$\Delta x = \frac{c^2 \Delta t}{v} \tag{80}$$

We analyze this expression functionally. Because the speed of light c and the time separation interval Δt are fixed physical constraints, the spatial interval Δx varies inversely with the relative frame velocity v:

$$\Delta x(v) \propto \frac{1}{v} \tag{81}$$

To make Δx as small as possible, we must make its denominator term as large as possible. Under the physical principles of Special Relativity, the relative speed v between two real inertial reference frames is strictly bounded by the speed of light in a vacuum:

$$v < c \implies \lim v \to c \tag{82}$$

Therefore, the absolute theoretical minimum spatial separation occurs at the asymptotic limit where the relative velocity approaches the speed of light:

$$\Delta x_{\min} = \lim_{v \to c} \frac{c^2 \Delta t}{v} = \frac{c^2 \Delta t}{c} = c\Delta t \tag{83}$$

Numerical Evaluation Substitute the given numerical parameters into the limiting equation ($c \approx 3.00 \times 10^8$ m/s and $\Delta t = 1.00 \times 10^{-9}$ s):

$$\Delta x_{\min} = (3.00 \times 10^8 \text{ m/s}) \times (1.00 \times 10^{-9} \text{ s}) \tag{84}$$

$$\Delta x_{\min} = 0.30 \text{ metres} \tag{85}$$

Final Analytical Result

The minimum possible spatial separation between the two events in frame S is:

$$\Delta x_{\min} = 0.3 \text{ m} \tag{86}$$

This occurs in the limit where the relative frame speed reaches its upper physical bound:

$$v \to c \quad (\text{or } v \approx 3 \times 10^8 \text{ m/s}) \tag{87}$$

Physics Hint: *This minimum bound can also be understood directly through the invariance of the spacetime interval:* $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2(\Delta t')^2 - (\Delta x')^2$. *Because* $\Delta t' = 0$, *we have* $c^2\Delta t^2 - \Delta x^2 = -(\Delta x')^2 \leq 0$, *which requires* $\Delta x^2 = c^2\Delta t^2 + (\Delta x')^2$. *Since* $(\Delta x')^2 > 0$ *for spatially separated events,* Δx *must always exceed* $c\Delta t$, *reaching its minimum when* $\Delta x' \to 0$.

Question 6

Problem Statement

6. (a) A Higgs boson (mass M_H) decays into two particles of equal mass m_q. Assuming that the particles travel with relativistic speeds, what are their energies and momenta in the Higgs boson rest frame? **[3]**

(b) Assume the Higgs boson mass = 125 GeV/c^2 and the two particles are b-quarks of mass = 5 GeV/c^2, what is the speed of each b-quark in the rest frame of the Higgs boson? (*Hint: consider the Lorentz factor* γ.) **[3]**

Complete Solution & Study Guide

(a) Decay Kinematics in the Rest Frame

Physical Setup and Symmetry Let us work entirely within the rest frame of the parent Higgs boson particle (denoted as frame S). By definition, the initial state consists of a single particle at rest:

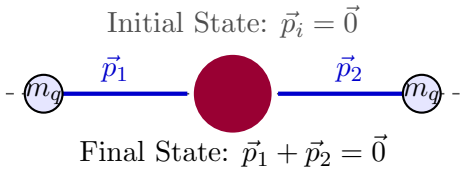
$$E_i = M_H c^2 \tag{88}$$

$$\vec{p}_i = \vec{0} \tag{89}$$

Following the decay event, we have two product particles (b-quarks) of identical rest mass $m_1 = m_2 = m_q$. Let their post-decay energies be E_1 and E_2, and their momentum vectors be $\vec{p}_1$ and $\vec{p}_2$.

Physics Hint: *In the rest frame of the decaying parent particle, total linear momentum is exactly zero. Due to conservation of momentum, the two product particles must fly apart in exactly opposite directions with equal momentum magnitudes. Because their masses are identical, their relativistic kinetic parameters and energies must also be perfectly symmetric.*

Higgs Boson Decoupling Geometry



Step-by-Step Algebraic Derivation We apply the Conservation of Total Linear Momentum:

$$\vec{p}_i = \vec{p}_f \implies \vec{0} = \vec{p}_1 + \vec{p}_2 \implies \vec{p}_1 = -\vec{p}_2 \tag{90}$$

Taking the absolute scalar magnitude of both sides shows that their momenta are equal:

$$|\vec{p}_1| = |\vec{p}_2| = p \tag{91}$$

Next, we write down the relativistic energy-momentum relation for each individual decay product:

$$E_1 = \sqrt{p_1^2c^2 + m_q^2c^4} = \sqrt{p^2c^2 + m_q^2c^4} \tag{92}$$

$$E_2 = \sqrt{p_2^2c^2 + m_q^2c^4} = \sqrt{p^2c^2 + m_q^2c^4} \tag{93}$$

This explicitly demonstrates that the energies of the two particles are equal:

$$E_1 = E_2 = E \tag{94}$$

Now, apply the Conservation of Total Relativistic Energy:

$$E_i = E_f \implies M_Hc^2 = E_1 + E_2 \tag{95}$$

Substitute the equality $E_1 = E_2 = E$ into the conservation law:

$$M_Hc^2 = 2E \tag{96}$$

$$E = \frac{M_Hc^2}{2} \tag{97}$$

To find the momentum magnitude p, rearrange the relativistic energy-momentum formula to isolate p^2c^2:

$$E^2 = p^2c^2 + m_q^2c^4 \implies p^2c^2 = E^2 - m_q^2c^4 \tag{98}$$

Substitute the expression $E = \frac{1}{2}M_Hc^2$ into this relation:

$$p^2c^2 = \left(\frac{M_Hc^2}{2}\right)^2 - m_q^2c^4 \tag{99}$$

$$p^2c^2 = \frac{1}{4}M_H^2c^4 - m_q^2c^4 = c^4\left(\frac{1}{4}M_H^2 - m_q^2\right) \tag{100}$$

Divide both sides by c^2 to solve for p^2:

$$p^2 = c^2\left(\frac{1}{4}M_H^2 - m_q^2\right) = \frac{c^2}{4}\left(M_H^2 - 4m_q^2\right) \tag{101}$$

Taking the positive square root gives the momentum magnitude of each particle:

$$p = \frac{c}{2}\sqrt{M_H^2 - 4m_q^2} \tag{102}$$

Thus, the energy and momentum of each particle in the rest frame are:

$$E = \frac{1}{2}M_Hc^2 \tag{103}$$

$$p = \frac{1}{2}c\sqrt{M_H^2 - 4m_q^2} \tag{104}$$

(b) Numerical Evaluation of Decay Velocity

Step 1: Calculating the Lorentz Factor γ We are given the following physical parameters for the particles:

- Parent Higgs boson mass: $M_H = 125 \text{ GeV}/c^2$
- Daughter b-quark mass: $m_q = 5 \text{ GeV}/c^2$

The total relativistic energy of a particle can be written in terms of its rest mass and its Lorentz factor γ:

$$E = \gamma m_q c^2 \tag{105}$$

From part (a), we know the exact energy of each b-quark is half the rest mass energy of the Higgs boson:

$$E = \frac{1}{2} M_H c^2 \tag{106}$$

Equating these two expressions for E:

$$\gamma m_q c^2 = \frac{1}{2} M_H c^2 \tag{107}$$

Cancel the common factor c^2 from both sides:

$$\gamma m_q = \frac{1}{2} M_H \implies \gamma = \frac{M_H}{2m_q} \tag{108}$$

Substitute the given numerical mass values into the equation:

$$\gamma = \frac{125 \text{ GeV}/c^2}{2 \times (5 \text{ GeV}/c^2)} = \frac{125}{10} = 12.5 \tag{109}$$

Physics Hint: *A Lorentz factor of $\gamma = 12.5$ is significantly greater than 1. This confirms that the b-quarks are highly relativistic, meaning their velocities are very close to the speed of light.*

Step 2: Deriving Speed from the Lorentz Factor By definition, the Lorentz factor relates to the particle speed v via:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{110}$$

To isolate v, square both sides of the equation:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \tag{111}$$

Take the reciprocal:

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} \tag{112}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \tag{113}$$

Taking the square root gives the speed as a fraction of the speed of light:

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} \tag{114}$$

Numerical Arithmetic Substitute the value $\gamma = 12.5$ into our velocity equation:

$$\frac{1}{\gamma^2} = \frac{1}{(12.5)^2} = \frac{1}{156.25} = 0.0064 \tag{115}$$

$$\frac{v}{c} = \sqrt{1 - 0.0064} = \sqrt{0.9936} \approx 0.99679 \tag{116}$$

Expressing the final velocity in terms of c:

$$v \approx 0.997c \quad (\text{or } 2.99 \times 10^8 \text{ m/s}) \tag{117}$$

Each b-quark travels at approximately 99.7% of the speed of light in the rest frame of the Higgs boson.

Question 7

Problem Statement

7. (a) A satellite of mass m orbits a star of mass M ($M \gg m$). Considering angular momentum, derive an expression for the total energy of the satellite, explaining each of the terms. **[6]**

(b) A small planet is in a circular orbit of radius R about a massive star. The star suddenly undergoes an explosion in which a small fraction ε of its mass is ejected spherically symmetrically to a large distance from the star (assume that none of the ejected mass hits the planet). Calculate the closest and furthest distances of the planet's orbit around the star after the explosion. **[14]**

Complete Solution & Study Guide

(a) Derivation of Total Energy in Terms of Angular Momentum

Coordinate Setup and Kinematics Consider a two-body system where a small satellite of mass m is bound gravitationally to a massive star of mass M. Because $M \gg m$, we assume the star remains stationary at the origin of a two-dimensional plane polar coordinate system (r, θ).

The position vector of the satellite is $\vec{r} = r\hat{r}$. Differentiating with respect to time gives the velocity vector in polar coordinates:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{118}$$

where $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}$ represents the radial velocity component, and $r\dot{\theta} = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$ represents the transverse (azimuthal) velocity component.

The total mechanical energy E of the satellite is the sum of its kinetic energy K and its gravitational potential energy U:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{119}$$

Substitute the magnitude square of the velocity vector $v^2 = \vec{v} \cdot \vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r} \tag{120}$$

Elimination of the Angular Parameter The gravitational force is purely radial, meaning it exerts no torque ($\vec{\tau} = \vec{r} \times \vec{F} = \vec{0}$) about the central origin. Consequently, the orbital angular momentum vector $\vec{L}$ is a constant of motion. Its magnitude is given by:

$$L = |\vec{r} \times m\vec{v}| = |(r\hat{r}) \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})| = mr^2\dot{\theta} \tag{121}$$

We isolate the angular velocity parameter $\dot{\theta}$ from this conservation relationship:

$$\dot{\theta} = \frac{L}{mr^2} \tag{122}$$

Substitute this expression back into the transverse kinetic energy term of our total energy equation:

$$K_{\text{transverse}} = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}mr^2\left(\frac{L}{mr^2}\right)^2 = \frac{1}{2}mr^2\left(\frac{L^2}{m^2r^4}\right) = \frac{L^2}{2mr^2} \tag{123}$$

Re-assembling the total energy equation yields:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{124}$$

Explanation of Terms within the Effective Potential We can group the two structural terms that depend solely on the spatial radial coordinate r into a single function termed the **effective potential energy**, $V_{\text{eff}}(r)$:

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r), \quad \text{where } V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{125}$$

1. $\mathbf{\frac{1}{2}m\dot{r}^2}$: This represents the **radial kinetic energy** of the satellite, associated with moving directly toward or away from the central star.
2. $\mathbf{\frac{L^2}{2mr^2}}$: This is the **centrifugal potential energy barrier**. It is a pseudo-potential term arising from the transformation into polar coordinates. Because it scales as $+1/r^2$, it acts as a strong repulsive barrier at short distances, physically preventing a particle with non-zero angular momentum from crashing into the central origin.
3. $\mathbf{-\frac{GMm}{r}}$: This is the standard attractive **Newtonian gravitational potential energy** between two point masses, establishing the attractive bound well that pulls the satellite toward the star.

Physics Hint: *Grouping the terms into an effective potential $V_{eff}(r)$ maps a complex two-dimensional orbital problem onto an equivalent one-dimensional system governed by an effective force $F_{eff} = -\frac{dV_{eff}}{dr}$.*

(b) Mass Ejection and Post-Explosion Apsidal Distances

Step 1: Analyzing the Initial State (Pre-Explosion) Initially, the small planet describes a perfect circular orbit of radius R around the star of mass M. For a circular orbit, the radial coordinate is strictly constant

($\dot{r} = 0$). The gravitational attraction balances the required centripetal acceleration exactly:

$$\frac{GMm}{R^2} = \frac{mv_{\text{circ}}^2}{R} \implies v_{\text{circ}}^2 = \frac{GM}{R} \tag{126}$$

The initial angular momentum magnitude L can be expressed as:

$$L = mRv_{\text{circ}} \implies L^2 = m^2R^2v_{\text{circ}}^2 = m^2R^2\left(\frac{GM}{R}\right) = GMm^2R \tag{127}$$

Step 2: Defining parameters Immediately After Explosion The problem states that a small fraction ε of the star's mass is ejected *suddenly* and *spherically symmetrically.*

- **Spherically Symmetric Ejection**: Shells of matter expanding symmetrically outward exert zero net gravitational force on objects inside them, according to Newton's shell theorem. Once the shell expands past the planet's orbital radius R, the mass enclosed inside the planet's orbit drops instantly from M to $M_f = M(1-\varepsilon)$.
- **Sudden Approximation**: The explosion happens so rapidly that the planet does not have time to alter its spatial position vector during the event. Thus, immediately after the explosion, the planet remains at radius $r = R$ and retains its original instantaneous velocity vector magnitude $v = v_{\text{circ}}$.

Consequently, the planet's angular momentum L is conserved across the transition because no non-radial forces act on it:

$$L_f^2 = L^2 = GMm^2R \tag{128}$$

However, because the central gravitating mass has changed to $M_f = M(1-\varepsilon)$, the total energy must be re-evaluated with the new potential energy parameters:

$$E_f = K_f + U_f = \frac{1}{2}mv_{\text{circ}}^2 - \frac{GM(1-\varepsilon)m}{R} \tag{129}$$

Substitute our known circular velocity value $v_{\text{circ}}^2 = \frac{GM}{R}$:

$$E_f = \frac{1}{2}m\left(\frac{GM}{R}\right) - \frac{GMm}{R} + \frac{GM\varepsilon m}{R} \tag{130}$$

$$E_f = \frac{GMm}{R}\left(\frac{1}{2} - 1 + \varepsilon\right) = \frac{GMm}{R}\left(\varepsilon - \frac{1}{2}\right) \tag{131}$$

Physics Hint: *Notice that for the orbit to remain bound ($E_f < 0$), we must have $\varepsilon < 1/2$. If the star loses more than half its mass, the energy becomes positive, and the planet will escape to infinity along a parabolic or hyperbolic trajectory.*

Step 3: Calculating Turning Points (Apsidal Distances) The turning points of the new elliptical orbit (the closest distance $r_{\min}$ and furthest distance $r_{\max}$) occur when the radial velocity component momentarily vanishes ($\dot{r} = 0$). Setting $\dot{r} = 0$ in our post-explosion total energy framework:

$$E_f = \frac{L_f^2}{2mr^2} - \frac{GM_f m}{r} \tag{132}$$

Substitute the known post-explosion parameters $E_f = \frac{GMm}{R}\left(\varepsilon - \frac{1}{2}\right)$, $L_f^2 = GMm^2R$, and $M_f = M(1-\varepsilon)$:

$$\frac{GMm}{R}\left(\varepsilon - \frac{1}{2}\right) = \frac{GMm^2R}{2mr^2} - \frac{GM(1-\varepsilon)m}{r} \tag{133}$$

We simplify this expression by dividing every term by the common scalar factor GMm:

$$\frac{1}{R}\left(\varepsilon - \frac{1}{2}\right) = \frac{R}{2r^2} - \frac{1-\varepsilon}{r} \tag{134}$$

To eliminate fractions, multiply the entire equation by the common denominator $2r^2R$:

$$2r^2\left(\varepsilon - \frac{1}{2}\right) = R^2 - 2rR(1-\varepsilon) \tag{135}$$

$$(2\varepsilon - 1)r^2 + 2R(1-\varepsilon)r - R^2 = 0 \tag{136}$$

Multiply the entire equation by -1 to establish a standard quadratic equation format $ar^2 + br + c = 0$:

$$(1-2\varepsilon)r^2 - 2R(1-\varepsilon)r + R^2 = 0 \tag{137}$$

Step 4: Solving the Quadratic Equation We apply the quadratic formula $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ with coefficients:

$$\begin{aligned} a &= 1 - 2\varepsilon \\ b &= -2R(1-\varepsilon) \\ c &= R^2 \end{aligned}$$

Let us evaluate the discriminant term $(b^2 - 4ac)$ first:

$$\begin{aligned} b^2 - 4ac &= [-2R(1-\varepsilon)]^2 - 4(1-2\varepsilon)R^2 \\ &= 4R^2(1 - 2\varepsilon + \varepsilon^2) - 4R^2(1-2\varepsilon) \\ &= 4R^2 - 8R^2\varepsilon + 4R^2\varepsilon^2 - 4R^2 + 8R^2\varepsilon \\ &= 4R^2\varepsilon^2 \end{aligned}$$

Taking the square root of the simplified discriminant:

$$\sqrt{b^2 - 4ac} = \sqrt{4R^2\varepsilon^2} = 2R\varepsilon \tag{138}$$

Substitute these values back into the quadratic formula expression:

$$r = \frac{2R(1-\varepsilon) \pm 2R\varepsilon}{2(1-2\varepsilon)} \tag{139}$$

Factor out and cancel the multiplier 2 from both the numerator and denominator:

$$r = \frac{R(1-\varepsilon) \pm R\varepsilon}{1-2\varepsilon} \tag{140}$$

Let us analyze the two distinct mathematical roots separately:

1. **The Minus Root (Closest Distance, $r_{\mathbf{min}}$)**:

$$r_- = \frac{R(1-\varepsilon) - R\varepsilon}{1-2\varepsilon} = \frac{R - R\varepsilon - R\varepsilon}{1-2\varepsilon} = \frac{R(1-2\varepsilon)}{1-2\varepsilon} = R \tag{141}$$

2. **The Plus Root (Furthest Distance, $r_{\mathbf{max}}$)**:

$$r_+ = \frac{R(1-\varepsilon) + R\varepsilon}{1-2\varepsilon} = \frac{R - R\varepsilon + R\varepsilon}{1-2\varepsilon} = \frac{R}{1-2\varepsilon} \tag{142}$$

Final Analytical Interpretation

- **Closest Distance (Periapsis)**:

$$r_{\text{min}} = R \tag{143}$$

- **Furthest Distance (Apoapsis)**:

$$r_{\text{max}} = \frac{R}{1-2\varepsilon} \tag{144}$$

Physics Hint: *The result $r_{min} = R$ makes perfect physical sense. Since the explosion was instantaneous, the planet was at radius R with its velocity vector still exactly perpendicular to its position vector ($\vec{r} \cdot \vec{v} = 0$). This means the starting point remains an apsis of the new orbit. Because the star lost mass, the gravitational pull weakened, causing the planet to swing outward. Thus, the initial radius R becomes the minimum point (periapsis) of the new, expanded elliptical path.*

Question 8

Problem Statement

8. Define the invariant interval in space-time. Starting with the Lorentz transformations, verify that the invariant interval is indeed invariant with change of inertial reference frame.

Hence, or otherwise, calculate the factor by which time is dilated for observers in two inertial reference frames moving at a constant relative speed.

An astronaut sitting at the back of a rocket of proper length l moving at speed $0.5c$ away from Earth throws a ball to the front of the rocket at speed $0.8c$ relative to the rocket. What distance does the ball travel, and how long does it take, according to:

(a) the astronaut

(b) a stationary observer at Mission Control on Earth

(c) a massless flea resting on the ball.

According to Mission Control, one year has passed since the rocket left Earth with speed $0.5c$; how much time has elapsed in the frame of the astronaut? **[9]** **[10]** **[1]**

Complete Solution & Study Guide

Definitions & Space-Time Invariance Proof

Definition of the Invariant Interval In Special Relativity, the differential space-time interval $\mathrm{d}s^2$ (or its finite counterpart Δs^2 for flat Minkowski space) between two events separated by a spatial distance Δx and a temporal displacement Δt along a single spatial dimension is defined as:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 \tag{145}$$

Physics Hint: *The sign convention chosen above $(+,-,-,-)$ is known as the timelike-positive convention. An alternative convention is available $(-,+,+,+)$, which yields $\Delta s^2 = \Delta x^2 - c^2\Delta t^2$. Both frameworks are physically identical as long as consistency is strictly maintained throughout the calculations.*

Mathematical Verification via Lorentz Transformations Let frame S' move with constant speed v along the positive x-axis relative to frame S. The coordinate intervals between two events map via the standard linear differential Lorentz transformations:

$$\Delta x' = \gamma(\Delta x - v\Delta t) \tag{146}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \tag{147}$$

where the dimensionless scaling factor is $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$.

To test the invariance of the space-time interval, we formulate the expression for the interval $\Delta s'^2$ in the moving frame S' and substitute these transformation equations:

$$\Delta s'^2 = c^2(\Delta t')^2 - (\Delta x')^2 \tag{148}$$

$$\Delta s'^2 = c^2\left[\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)\right]^2 - [\gamma(\Delta x - v\Delta t)]^2 \tag{149}$$

Factor out the common scalar factor γ^2:

$$\Delta s'^2 = \gamma^2\left\{c^2\left[\Delta t^2 - \frac{2v\Delta x\Delta t}{c^2} + \frac{v^2\Delta x^2}{c^4}\right] - \left[\Delta x^2 - 2v\Delta x\Delta t + v^2\Delta t^2\right]\right\} \tag{150}$$

Distribute the constants c^2 and the negative sign carefully:

$$\Delta s'^2 = \gamma^2\left\{c^2\Delta t^2 - 2v\Delta x\Delta t + \frac{v^2\Delta x^2}{c^2} - \Delta x^2 + 2v\Delta x\Delta t - v^2\Delta t^2\right\} \tag{151}$$

Notice that the cross-multiplied tracking terms $-2v\Delta x\Delta t$ and $+2v\Delta x\Delta t$ cancel each other out:

$$\Delta s'^2 = \gamma^2\left\{c^2\Delta t^2 - v^2\Delta t^2 - \Delta x^2 + \frac{v^2\Delta x^2}{c^2}\right\} \tag{152}$$

Group the remaining expressions by grouping common factors of Δt^2 and Δx^2:

$$\Delta s'^2 = \gamma^2\left\{\Delta t^2(c^2 - v^2) - \Delta x^2\left(1 - \frac{v^2}{c^2}\right)\right\} \tag{153}$$

Pull out a factor of c^2 from the first bracketed grouping:

$$\Delta s'^2 = \gamma^2\left\{c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) - \Delta x^2\left(1 - \frac{v^2}{c^2}\right)\right\} \tag{154}$$

Factor out the shared polynomial term $\left(1 - \frac{v^2}{c^2}\right)$:

$$\Delta s'^2 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)\left[c^2\Delta t^2 - \Delta x^2\right] \tag{155}$$

Recall the algebraic definition of the Lorentz parameter: $\gamma^2 = \frac{1}{1-v^2/c^2}$. Therefore, the product simplifies to unity:

$$\gamma^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{1}{1 - \frac{v^2}{c^2}} \cdot \left(1 - \frac{v^2}{c^2}\right) = 1 \tag{156}$$

This leaves us with:

$$\Delta s'^2 = c^2 \Delta t^2 - \Delta x^2 = \Delta s^2 \tag{157}$$

The interval value is preserved exactly, verifying that the space-time interval is a Lorentz invariant scalar.

Derivation of the Time Dilation Factor

Let an observer carry a clock that remains permanently at rest in their own reference frame S'. This observer measures the time elapsed between two ticks occurring at the same position. Because the clock does not translate in this frame, its spatial tracking displacement interval is zero:

$$\Delta x' = 0 \tag{158}$$

The time interval measured on a clock at rest is defined as the **proper time**, denoted $\Delta\tau = \Delta t'$.

We compute the corresponding interval seen by an observer in frame S, where the clock is observed moving at speed v. Using the proven property of space-time interval invariance ($\Delta s^2 = \Delta s'^2$):

$$c^2 \Delta t^2 - \Delta x^2 = c^2 (\Delta t')^2 - (\Delta x')^2 \tag{159}$$

Substitute the rest conditions $\Delta x' = 0$ and $\Delta t' = \Delta\tau$:

$$c^2 \Delta t^2 - \Delta x^2 = c^2 (\Delta\tau)^2 \tag{160}$$

In frame S, the clock moves at constant speed v, covering a spatial distance of $\Delta x = v\Delta t$ over the course of the time interval. Substituting this kinematic path profile:

$$c^2 \Delta t^2 - (v\Delta t)^2 = c^2 \Delta\tau^2 \tag{161}$$

$$\Delta t^2 (c^2 - v^2) = c^2 \Delta\tau^2 \tag{162}$$

Divide both sides by c^2:

$$\Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = \Delta\tau^2 \tag{163}$$

Taking the square root isolates the time dilation relationship:

$$\Delta t \sqrt{1 - \frac{v^2}{c^2}} = \Delta\tau \implies \Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta\tau \tag{164}$$

The time dilation factor is the Lorentz factor γ.

Kinematics of the Thrown Ball

We define three distinct inertial reference frames:

- Frame S: Earth / Mission Control base frame.
- Frame S': The rocket frame, translating at speed $v = 0.5c$ relative to Earth.
- Frame S'': The ball frame, moving at speed $u' = 0.8c$ relative to the rocket.

(a) Calculations according to the Astronaut (Rocket Frame S') The astronaut resides at rest inside the rocket frame S'. The physical length of the rocket measured in this frame is its proper length, given as l. The ball is thrown from the back of the rocket ($x'_1 = 0$) to the front ($x'_2 = l$). Therefore, the distance traversed by the ball according to the astronaut is simply:

$$\Delta x' = l \tag{165}$$

The ball travels at a constant relative speed of $u' = 0.8c$ within this frame. The time required to span the distance is given by basic kinematics:

$$\Delta t' = \frac{\Delta x'}{u'} = \frac{l}{0.8c} = 1.25\frac{l}{c} \tag{166}$$

(b) Calculations according to Mission Control (Earth Frame S) To compute the velocity of the ball relative to Earth, u, we apply the relativistic velocity addition formula, combining the rocket speed $v = 0.5c$ and the ball's relative speed $u' = 0.8c$:

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{0.8c + 0.5c}{1 + \frac{(0.8c)(0.5c)}{c^2}} = \frac{1.3c}{1 + 0.40} = \frac{1.3c}{1.4} = \frac{13}{14}c \approx 0.9286c \tag{167}$$

To compute the flight parameters Δx and Δt observed by Earth, we map the rocket frame endpoints ($\Delta x' = l, \Delta t' = 1.25\frac{l}{c}$) into frame S using the inverse Lorentz transformations:

$$\gamma_{\text{rocket}} = \frac{1}{\sqrt{1 - (0.5)^2}} = \frac{1}{\sqrt{0.75}} = \frac{2}{\sqrt{3}} \approx 1.1547 \tag{168}$$

Evaluating the time elapsed according to Mission Control:

$$\Delta t = \gamma_{\text{rocket}} \left(\Delta t' + \frac{v\Delta x'}{c^2} \right) = \frac{2}{\sqrt{3}} \left(1.25\frac{l}{c} + \frac{0.5c \cdot l}{c^2} \right) \tag{169}$$

$$\Delta t = \frac{2}{\sqrt{3}} \left(1.25\frac{l}{c} + 0.5\frac{l}{c} \right) = \frac{2}{\sqrt{3}} \left(1.75\frac{l}{c} \right) = \frac{3.5}{\sqrt{3}}\frac{l}{c} = \frac{7}{2\sqrt{3}}\frac{l}{c} \approx 2.0207\frac{l}{c} \tag{170}$$

Evaluating the distance traveled according to Mission Control:

$$\Delta x = \gamma_{\text{rocket}} \left(\Delta x' + v\Delta t' \right) = \frac{2}{\sqrt{3}} \left(l + 0.5c \cdot \left(1.25 \frac{l}{c} \right) \right) \tag{171}$$

$$\Delta x = \frac{2}{\sqrt{3}} \left(l + 0.625l \right) = \frac{2}{\sqrt{3}} \left(1.625l \right) = \frac{3.25}{\sqrt{3}} l = \frac{13}{4\sqrt{3}} l \approx 1.8764l \tag{172}$$

Physics Hint: *We check consistency using $\Delta x = u\Delta t$: $\left(\frac{13}{14}c\right) \times \left(\frac{7}{2\sqrt{3}} \frac{l}{c}\right) = \frac{13 \times 7}{14 \times 2\sqrt{3}} l = \frac{13}{4\sqrt{3}} l$. This matches our transformation result, confirming the numerical calculations are correct.*

(c) Calculations according to the Flea (Ball Rest Frame S'') The flea is at rest relative to the ball, so it sits inside frame S''. By definition, the ball does not translate within its own rest frame:

$$\Delta x'' = 0 \tag{173}$$

The time elapsed in the ball's rest frame is its proper time interval $\Delta t''$. We calculate this value by invoking the invariance of the space-time interval between the rocket frame S' and the ball frame S'':

$$c^2 (\Delta t'')^2 - (\Delta x'')^2 = c^2 (\Delta t')^2 - (\Delta x')^2 \tag{174}$$

Substitute $\Delta x'' = 0$, $\Delta x' = l$, and $\Delta t' = 1.25 \frac{l}{c}$:

$$c^2 (\Delta t'')^2 - 0 = c^2 \left(1.25 \frac{l}{c} \right)^2 - l^2 = 1.5625 l^2 - l^2 = 0.5625 l^2 \tag{175}$$

$$(\Delta t'')^2 = 0.5625 \frac{l^2}{c^2} \implies \Delta t'' = \sqrt{0.5625} \frac{l}{c} = 0.75 \frac{l}{c} = \frac{3}{4} \frac{l}{c} \tag{176}$$

Long-Term Journey Elapsed Time

The problem states that according to Mission Control (Earth frame S), a total time interval of $\Delta T_{\text{Earth}} = 1$ year has passed since departure. We want to find the corresponding elapsed time interval ΔT_{astro} in the rocket frame.

The astronaut remains at rest relative to the rocket, meaning their clock measures proper time. We apply the direct time dilation relationship between the rocket and Earth:

$$\Delta T_{\text{Earth}} = \gamma_{\text{rocket}} \Delta T_{\text{astro}} \tag{177}$$

Substitute the values $\Delta T_{\text{Earth}} = 1$ year and $\gamma_{\text{rocket}} = \frac{2}{\sqrt{3}}$:

$$1 \text{ year} = \frac{2}{\sqrt{3}} \Delta T_{\text{astro}} \tag{178}$$

$$\Delta T_{\text{astro}} = \frac{\sqrt{3}}{2} \text{ years} \approx 0.866 \text{ years} \tag{179}$$

Thus, approximately 0.866 years (or about 10.4 months) have elapsed for the astronaut.

Question 9

Problem Statement

9. State Lagrange's equations of motion for a system described by a Lagrangian $L(q_i, \dot{q}_i, t)$, where q_i are generalised coordinates and $\dot{q}_i$ are generalised velocities. Define what is meant by a cyclic coordinate and explain how it is related to a conservation law. **[3]**

A uniform thin rod of length L and negligible mass is supported by a frictionless pivot at one end, and has a mass M glued to the other end, such that it is free to swing in a fixed vertical plane. Write down the Lagrangian of the system. Hence, or otherwise, find the period of small oscillations of the rod in this vertical plane. **[8]**

Suppose the support is changed to allow the rod to swing in any direction. Using suitable generalised coordinates, write down the Lagrangian of the system and hence identify the cyclic coordinate and the respective conserved quantity. **[5]**

Write down the Hamiltonian of the system and hence state whether the energy is a constant of motion. **[4]**

Complete Solution & Study Guide

Fundamental Definitions & Cyclic Coordinates

Lagrange's Equations of Motion For a conservative mechanical system with N degrees of freedom characterized by independent generalized coordinates q_i and generalized velocities $\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t}$ (where $i = 1, 2, \ldots, N$), the system's dynamics are completely determined by the Lagrangian function $L = T - U$. Here, T represents total kinetic energy and U represents total potential energy. The equations of motion are given by:

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \text{for } i = 1, 2, \ldots, N \tag{180}$$

Cyclic Coordinates and Conservation Laws A generalized coordinate q_k is defined to be **cyclic** (or ignorable) if it does not explicitly appear anywhere in the system's Lagrangian function, meaning:

$$\frac{\partial L}{\partial q_k} = 0 \tag{181}$$

When a coordinate is cyclic, Lagrange's equation for that specific coordinate index k simplifies immediately:

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \tag{182}$$

This implies that the quantity enclosed in parentheses must be constant over time. This quantity is defined as the generalized canonical momentum p_k conjugate to the coordinate q_k:

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant of motion} \tag{183}$$

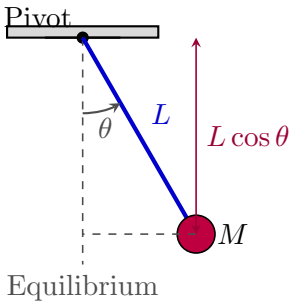
Thus, every cyclic coordinate directly corresponds to a conserved conjugate momentum. This is a primary manifestation of Noether's Theorem relating spatial symmetries to physical conservation laws.

Physics Hint: *If a Lagrangian is invariant under shifts in a coordinate q_k (implying translational or rotational symmetry), its conjugate momentum is conserved. For example, if L is independent of an angle ϕ, the component of angular momentum along that axis remains constant throughout the motion.*

Planar Pendulum: Lagrangian and Small Oscillations

Physical Setup & Lagrangian Construction The system consists of a rigid, mass-less rod of length L fixed to a frictionless pivot at one end. A point particle of mass M is attached to the opposite end. The motion is restricted to a single vertical plane. We choose the single generalized coordinate to be the deflection angle θ measured relative to the downward vertical equilibrium line.

Geometry of the Planar Pendulum



The velocity magnitude of the mass bob is given by $v = L\dot{\theta}$. The kinetic energy T is:

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}ML^2\dot{\theta}^2 \tag{184}$$

We define our potential energy reference level ($U = 0$) at the horizontal line passing through the pivot. The vertical height of the mass below the

pivot is $-L\cos\theta$. Therefore, the gravitational potential energy U relative to this reference level is:

$$U = -MgL\cos\theta \tag{185}$$

Alternatively, referencing $U = 0$ at the lowest bottom turning point gives $U = MgL(1-\cos\theta)$. Using this setup:

$$L = T - U = \frac{1}{2}ML^2\dot{\theta}^2 - MgL(1-\cos\theta) \tag{186}$$

Deriving the Equation of Motion We compute the partial derivatives of the Lagrangian with respect to θ and $\dot{\theta}$:

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\big[-MgL + MgL\cos\theta\big] = -MgL\sin\theta \tag{187}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}ML^2\dot{\theta}^2\right] = ML^2\dot{\theta} \tag{188}$$

Now, evaluate the total time derivative of the generalized canonical momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\big(ML^2\dot{\theta}\big) = ML^2\ddot{\theta} \tag{189}$$

Substituting these components into Lagrange's equation:

$$-MgL\sin\theta - ML^2\ddot{\theta} = 0 \implies ML^2\ddot{\theta} + MgL\sin\theta = 0 \tag{190}$$

Divide out the common constant terms ML to isolate the acceleration term:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{191}$$

Small Oscillations Approximation For small angular displacements ($\theta \ll 1$), we use the first-order Taylor series approximation for the sine function: $\sin\theta \approx \theta$. Substituting this yields a standard second-order differential equation for a simple harmonic oscillator:

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{192}$$

This matches the standard form $\ddot{\theta}+\omega^2\theta = 0$, where the angular frequency is:

$$\omega = \sqrt{\frac{g}{L}} \tag{193}$$

The cyclic period of oscillation P is related to the angular frequency by $P = \frac{2\pi}{\omega}$:

$$P = \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\sqrt{\frac{L}{g}} \tag{194}$$

Spherical Pendulum: 3D Angular Dynamics

Generalized Coordinates and Kinetic Profile If the support pivot allows the rod to swing freely in any direction in three dimensions, the system has two degrees of freedom. We adopt standard spherical coordinates (r, θ, ϕ) centered on the pivot. Because the rod is rigid, the radial coordinate is fixed ($r = L$, $\dot{r} = 0$). The two remaining independent generalized coordinates are:

- θ: The polar (colatitude) angle measured relative to the downward vertical axis.
- ϕ: The azimuthal angle measured around the vertical axis.

The velocity squared magnitude in spherical polar coordinates under the constraint $r = L$ is:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = L^2\dot{\theta}^2 + L^2\sin^2\theta\dot{\phi}^2 \tag{195}$$

The kinetic energy function T is:

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2 \tag{196}$$

The potential energy depends only on the vertical position, which is entirely determined by the polar angle θ:

$$U = MgL(1 - \cos\theta) \tag{197}$$

Assembling the new system Lagrangian $L = T - U$:

$$L(\theta, \dot{\theta}, \dot{\phi}) = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2 - MgL(1 - \cos\theta) \tag{198}$$

Identification of Symmetries and Cyclic Quantities We check for explicit dependence on the coordinates θ and ϕ:

$$\frac{\partial L}{\partial \theta} = ML^2\sin\theta\cos\theta\dot{\phi}^2 - MgL\sin\theta \neq 0 \tag{199}$$

$$\frac{\partial L}{\partial \phi} = 0 \tag{200}$$

Because the coordinate ϕ does not appear anywhere in the expression for L, **the azimuthal angle ϕ is the cyclic coordinate**.

The conserved quantity is the canonical momentum p_ϕ conjugate to this cyclic coordinate:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left[\frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2\right] = ML^2\sin^2\theta\dot{\phi} \tag{201}$$

$$p_\phi = ML^2 \sin^2\theta\dot{\phi} = \text{constant of motion} \tag{202}$$

Physically, p_ϕ represents the component of the orbital angular momentum vector parallel to the vertical z-axis (L_z). This component is conserved due to the system's rotational symmetry about the vertical axis.

Hamiltonian Formulation

The Hamiltonian function H is defined by performing a Legendre transformation on the Lagrangian function:

$$H = \sum_i p_i\dot{q}_i - L = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L \tag{203}$$

where the generalized conjugate momenta are defined as:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ML^2\dot{\theta} \implies \dot{\theta} = \frac{p_\theta}{ML^2} \tag{204}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ML^2\sin^2\theta\dot{\phi} \implies \dot{\phi} = \frac{p_\phi}{ML^2\sin^2\theta} \tag{205}$$

Substitute these momentum expressions back into the Legendre transformation equation:

$$H = (ML^2\dot{\theta})\dot{\theta} + (ML^2\sin^2\theta\dot{\phi})\dot{\phi} - \left[\frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2 - U(\theta)\right] \tag{206}$$

$$H = ML^2\dot{\theta}^2 + ML^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}ML^2\dot{\theta}^2 - \frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2 + U(\theta) \tag{207}$$

$$H = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}ML^2\sin^2\theta\dot{\phi}^2 + U(\theta) = T + U \tag{208}$$

Expressing H in terms of the canonical momenta coordinates:

$$H(2\theta, p_\theta, p_\phi) = \frac{p_\theta^2}{2ML^2} + \frac{p_\phi^2}{2ML^2\sin^2\theta} + MgL(1-\cos\theta) \tag{209}$$

Conservation of Energy Analysis We assess whether the Hamiltonian is a constant of motion by calculating its total time derivative using Hamilton's equations:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{210}$$

Looking closely at our expression for H (or L), we see there is no explicit dependence on the time variable t:

$$\frac{\partial L}{\partial t} = 0 \implies \frac{\partial H}{\partial t} = 0 \implies \frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{211}$$

Since $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$, the Hamiltonian value is strictly constant over time. Furthermore, because our generalized coordinate definitions do not depend explicitly on time ($\frac{\partial \vec{r}}{\partial t} = \vec{0}$), the kinetic energy is a homogeneous quadratic function of the generalized velocities. This guarantees that the Hamiltonian equals the total mechanical energy ($H = T + U$). Therefore, the total energy of the system is a **constant of motion**.

Question 10

Problem Statement

10. Draw a simple illustrative diagram and state three advantages of using the centre-of-mass frame when considering the collision of two particles moving with non-relativistic speeds. **[4]**

Two particles A (of mass m) and B (of mass $M > m$) undergo a head-on elastic collision. Before the collision, particle A has speed u ($u \ll c$) and particle B is at rest. Calculate the speeds of A and B after the collision in:

a) the centre-of-mass frame **[6]**

b) the laboratory frame in which B was initially at rest. **[4]**

If instead the collision is not head-on, and particle A is deflected through an angle of 30°, calculate the final speeds of A and B in the laboratory frame for the case $M = 2m$. **[6]**

Complete Solution & Study Guide

The Centre-of-Mass Frame Overview

The centre-of-mass (CM) frame, or zero-momentum frame, is an inertial reference frame whose origin moves at the velocity of the system's centre of mass.

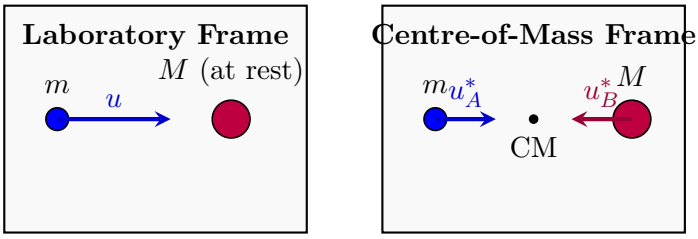
Three Advantages of the CM Frame

1. **Vanishing Total Momentum**: By definition, the total linear momentum is exactly zero ($\vec{P}^*_{\text{total}} = \vec{0}$). This means the two interacting particles always possess equal and opposite momentum vectors ($\vec{p}^*_A = -\vec{p}^*_B$) both before and after the collision, significantly simplifying the mathematical vector bookkeeping.

2. **Simplified Energy Allocation**: The total kinetic energy in the CM frame can be elegantly expressed using a single parameter—the reduced mass $\mu = \frac{mM}{m+M}$—via $K^* = \frac{p^{*2}}{2\mu}$. In an elastic collision, the magnitude of each particle's momentum is strictly conserved ($p^*_{\text{initial}} = p^*_{\text{final}}$).

3. **Kinematic Decoupling**: The complicated scattering dynamics are reduced to a simple re-orientation of the relative velocity vector. The scattering angle in the CM frame completely dictates the final state, separating the intrinsic internal collision physics from the overall translational motion of the system.

Kinematics in Laboratory and Centre-of-Mass Frames



Head-on Collision Analytics

Velocity of the Centre of Mass In the Laboratory frame, particle A has mass m and initial velocity $v_{A,i} = u$. Particle B has mass M and is stationary ($v_{B,i} = 0$). The velocity of the centre of mass (v_{CM}) relative to the Lab frame is:

$$v_{\text{CM}} = \frac{mv_{A,i} + Mv_{B,i}}{m+M} = \frac{mu + M(0)}{m+M} = \frac{mu}{m+M} \tag{212}$$

a) Post-Collision Speeds in the CM Frame To transform initial velocities from the Lab frame to the CM frame, subtract v_{CM}:

$$u_A^* = v_{A,i} - v_{\text{CM}} = u - \frac{mu}{m+M} = \frac{(m+M)u - mu}{m+M} = \frac{Mu}{m+M} \tag{213}$$

$$u_B^* = v_{B,i} - v_{\text{CM}} = 0 - \frac{mu}{m+M} = -\frac{mu}{m+M} \tag{214}$$

Physics Hint: *In an elastic head-on collision within the CM frame, the total momentum must remain zero, and total kinetic energy must be conserved. The only kinematic solution that satisfies both conditions for a one-dimensional impact is for each particle to reverse its velocity direction while retaining its exact initial speed profile.*

Letting v_A^* and v_B^* represent the post-collision velocities in the CM frame:

$$v_A^* = -u_A^* = -\frac{Mu}{m+M} \tag{215}$$

$$v_B^* = -u_B^* = +\frac{mu}{m+M} \tag{216}$$

Taking the absolute values yields the post-collision speeds in the CM frame:

$$\text{Speed}_A^* = \frac{Mu}{m+M}, \qquad \text{Speed}_B^* = \frac{mu}{m+M} \tag{217}$$

b) Post-Collision Speeds in the Laboratory Frame To return to the Laboratory frame, we apply a Galilean boost, adding v_{CM} back to the post-collision CM velocities:

$$v_{A,f} = v_A^* + v_{\text{CM}} = -\frac{Mu}{m+M} + \frac{mu}{m+M} = \frac{(m-M)u}{m+M} \tag{218}$$

$$v_{B,f} = v_B^* + v_{\text{CM}} = \frac{mu}{m+M} + \frac{mu}{m+M} = \frac{2mu}{m+M} \tag{219}$$

Since $M > m$, the expression $(m-M)$ is negative, indicating particle A rebounds backward. Taking magnitudes to obtain speeds:

$$\text{Speed}_{A,\text{Lab}} = \frac{(M-m)u}{m+M}, \qquad \text{Speed}_{B,\text{Lab}} = \frac{2mu}{m+M} \tag{220}$$

Two-Dimensional Oblique Collision ($M = 2m$)

Step 1: Setting up the Mass Constraint Parameters Substitute $M = 2m$ into our general frame configuration expressions:

$$v_{\text{CM}} = \frac{mu}{m+2m} = \frac{u}{3} \tag{221}$$

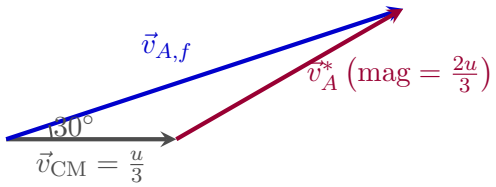
$$u_A^* = \frac{2mu}{m+2m} = \frac{2u}{3}, \qquad v_A^* = \frac{2u}{3} \tag{222}$$

$$u_B^* = -\frac{mu}{m+2m} = -\frac{u}{3}, \qquad v_B^* = \frac{u}{3} \tag{223}$$

Step 2: Vector Geometry of Particle A's Deflection In the Lab frame, particle A is deflected from its original trajectory by an angle $\theta = 30°$. The velocity vector in the Lab frame is the vector sum of its velocity in the CM frame and the velocity of the centre of mass: $\vec{v}_{A,f} = \vec{v}_A^* + \vec{v}_{\text{CM}}$.

We construct a vector triangle relating these three components. In this triangle, $\vec{v}_{A,f}$ forms an angle of 30° relative to the horizontal boost vector $\vec{v}_{\text{CM}}$. The magnitude of the CM velocity vector is constant: $|\vec{v}_A^*| = \frac{2u}{3}$.

Vector Addition Triangle for Particle A



Applying the Law of Cosines to this vector triangle:

$$|\vec{v}_A^*|^2 = |\vec{v}_{\text{CM}}|^2 + v_{A,f}^2 - 2v_{A,f}|\vec{v}_{\text{CM}}|\cos(30°) \tag{224}$$

Substitute the known magnitudes $|\vec{v}_A^*| = \frac{2u}{3}$ and $|\vec{v}_{\text{CM}}| = \frac{u}{3}$:

$$\left(\frac{2u}{3}\right)^2 = \left(\frac{u}{3}\right)^2 + v_{A,f}^2 - 2v_{A,f}\left(\frac{u}{3}\right)\left(\frac{\sqrt{3}}{2}\right) \tag{225}$$

$$\frac{4u^2}{9} = \frac{u^2}{9} + v_{A,f}^2 - \frac{\sqrt{3}u}{3}v_{A,f} \tag{226}$$

Rearranging this into a standard quadratic equation for $v_{A,f}$:

$$v_{A,f}^2 - \left(\frac{\sqrt{3}u}{3}\right)v_{A,f} - \frac{3u^2}{9} = 0 \implies v_{A,f}^2 - \left(\frac{u}{\sqrt{3}}\right)v_{A,f} - \frac{u^2}{3} = 0 \tag{227}$$

Applying the quadratic formula:

$$v_{A,f} = \frac{\frac{u}{\sqrt{3}} \pm \sqrt{\left(-\frac{u}{\sqrt{3}}\right)^2 - 4(1)\left(-\frac{u^2}{3}\right)}}{2} = \frac{\frac{u}{\sqrt{3}} \pm \sqrt{\frac{u^2}{3} + \frac{4u^2}{3}}}{2} \tag{228}$$

$$v_{A,f} = \frac{\frac{u}{\sqrt{3}} \pm \sqrt{\frac{5u^2}{3}}}{2} = \frac{u}{2\sqrt{3}}\left(1 \pm \sqrt{5}\right) \tag{229}$$

Since a speed must be strictly positive, we choose the positive root:

$$v_{A,f} = \frac{u}{2\sqrt{3}}\left(1 + \sqrt{5}\right) \approx 0.9341u \tag{230}$$

Step 3: Finding Particle B's Final Speed via Energy Conservation
Because the collision is perfectly elastic, total kinetic energy is conserved in the Lab frame. The initial kinetic energy equals the sum of the final kinetic energies of both particles:

$$K_i = K_f \implies \frac{1}{2}mu^2 = \frac{1}{2}mv_{A,f}^2 + \frac{1}{2}Mv_{B,f}^2 \tag{231}$$

Substitute the mass constraint $M = 2m$ and divide by $\frac{1}{2}m$:

$$u^2 = v_{A,f}^2 + 2v_{B,f}^2 \implies 2v_{B,f}^2 = u^2 - v_{A,f}^2 \tag{232}$$

We calculate $v_{A,f}^2$ using our exact radical expression:

$$v_{A,f}^2 = \left[\frac{u}{2\sqrt{3}}(1+\sqrt{5})\right]^2 = \frac{u^2}{12}(1+2\sqrt{5}+5) = \frac{u^2}{12}(6+2\sqrt{5}) = \frac{u^2}{6}(3+\sqrt{5}) \tag{233}$$

Substitute this back into the isolated energy expression to find $2v_{B,f}^2$:

$$2v_{B,f}^2 = u^2 - \frac{u^2}{6}(3+\sqrt{5}) = u^2\left(1 - \frac{3}{6} - \frac{\sqrt{5}}{6}\right) = u^2\left(\frac{1}{2} - \frac{\sqrt{5}}{6}\right) = \frac{u^2}{6}(3-\sqrt{5}) \tag{234}$$

Divide by 2 to solve for $v_{B,f}^2$:

$$v_{B,f}^2 = \frac{u^2}{12}(3 - \sqrt{5}) \tag{235}$$

Taking the square root gives the final speed of particle B:

$$v_{B,f} = \frac{u}{2\sqrt{3}}\sqrt{3 - \sqrt{5}} \approx 0.2523u \tag{236}$$

Final Analytical Speeds for Oblique Case

In the Laboratory frame, the final speeds are:

$$v_{A,f} = \frac{u}{2\sqrt{3}}\left(1 + \sqrt{5}\right) \approx 0.934u \tag{237}$$

$$v_{B,f} = \frac{u}{2\sqrt{3}}\sqrt{3 - \sqrt{5}} \approx 0.252u \tag{238}$$

Physics Hint: *We can double-check the final expression for $v_{B,f}$ by applying the radical identity $\sqrt{3 - \sqrt{5}} = \frac{\sqrt{5}-1}{\sqrt{2}}$, which gives another valid form for the speed: $v_{B,f} = \frac{u}{\sqrt{6}}\left(\frac{\sqrt{5}-1}{2}\right)$.*