A14119W1

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SHORT OPTIONS

TRINITY TERM 2023

Tuesday, 13 June

2:30 pm to 4:00 pm

Answer **two** *questions from the option for which you have entered.*

Start the answer to each question in a fresh book.

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

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Section S01 FUNCTIONS OF A COMPLEX VARIABLE

1. a) Prove that an analytic function must satisfy the Cauchy-Riemann conditions. [3]

b) Consider a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, such that

$$u(x, y) = \sum_{k=0}^{n/2} (-1)^k \frac{n!}{(2k)!(n-2k)!} x^{2k} y^{n-2k}$$

and n is an even, positive integer. Assume that $f = 0$ at $(x, y) = (0, 0)$ and find the v that makes $f(z)$ analytic. Express f as a function of z. [6]

c) Show that

$$\begin{aligned} w(z) &= (z - z_0)^n \\ z(w) &= (w - w_0)^{1/n} \end{aligned}$$

are conformal transformations. Sketch how they map the complex plane onto itself. [4]

d) Consider an infinite wedge with an opening angle π/m where m is a positive integer, with the vertex at $(1, 0)$ and one side lying on the the positive horizontal axis. The electrostatic potential, $\Phi(x, y)$ satisfies

$$\nabla^2 \Phi = 0$$

inside the wedge and the boundary conditions

$$\Phi(x, 0) = \Phi_0 \quad \text{and} \quad \Phi(x, (x-1)\tan\frac{\pi}{m}) = -\Phi_0$$

for $x \geq 1$. Use the conformal transformation described above to map the potential inside the wedge onto the half plane, $y \geq 0$. By considering an analytic function of the form $F = A \ln(we^{i\alpha})$ where A and α are constants, find the solution to the problem on the half plane. [6]

e) Find the solution to the problem within the wedge and sketch the lines of constant Φ and constant electric field $\vec{E} = -\vec{\nabla}\Phi$. [6]

2. a) Starting from Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

where $a = \frac{1}{3}e^{i\theta}$, choose an appropriate circular contour C (with radius $R > 1/3$) to calculate the following integral:

$$\int_0^{2\pi} \frac{e^{\cos\theta} \sin(\sin\theta)}{10 - 6\cos(\theta - \phi)} d\theta.$$

[Hint: it helps to evaluate $f(a) - f(R^2/a^*)$.] [7]

b) Find and classify all the singularities of

$$\frac{z}{e^{2z} - 1}.$$

Find all the corresponding residues. [6]

c) Find the Laurent expansion of

$$\exp\left(\frac{z}{z-3}\right)$$

around $z = 3$. [5]

d) Consider the following real integral

$$\int_0^{\infty} \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx.$$

Rewrite it as a contour integral, choose a contour and find its value. [7]

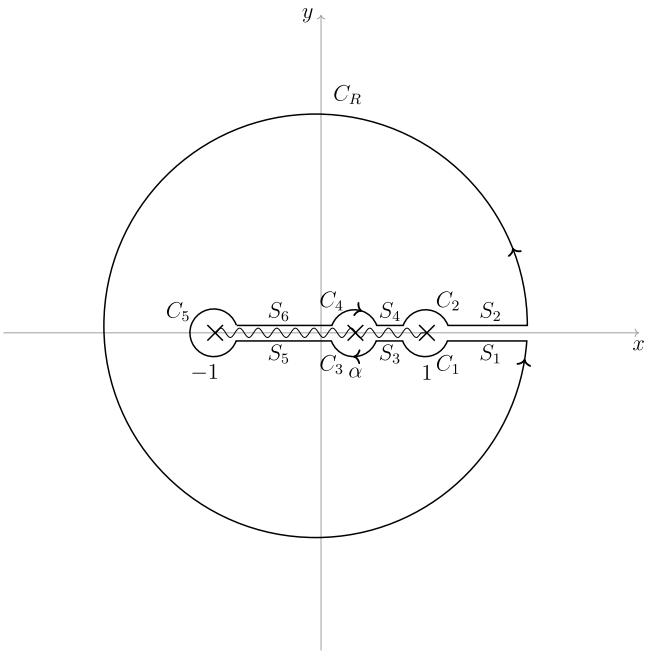
3. Consider the following function:

$$F(z) = \frac{\sqrt{z^2-1}}{\alpha - z}$$

where α is a real number.

a)Find all singularities and branch points and identify a branch cut. [4]

b) Consider the contour shown



where we have assumed $|\alpha| < 1$. Show that

$$\int_{C_R} F(z)dz = -i2\pi\alpha.$$

[5]

c) For $|\alpha| < 1$, show that

$$\int_{C_1 \cup C_2} F(z)dz = \int_{C_3 \cup C_4} F(z)dz = \int_{C_5} F(z)dz = 0.$$

[5]

d) For $|\alpha| < 1$, show that

$$I = \int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{\alpha - x} = \pi\alpha.$$

[4]

e) Now find I for $\alpha < -1$ and for $\alpha > 1$. [7]

Section S02 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. Polytropes are simple models of stars where the total pressure P at any stellar radius r is related to the volume mass density ρ at that radius by the relation

$$P(r) = \kappa\, \rho(r)^{\frac{n+1}{n}} \ ,$$

where κ is a constant and n is another constant called the polytropic index.

Consider a star where a constant fraction β $(0 < \beta < 1)$ of the total pressure is provided by an ideal gas and the rest by radiation. The pressure of an ideal gas is given by $P_{\rm g} = \frac{\rho}{\mu\, m_{\rm H}} k\, T$, where μ is the mean molecular mass, $m_{\rm H}$ is the mass of the Hydrogen atom, k is the Boltzman constant, and T is the temperature. The pressure of radiation is given by $P_{\rm r} = \frac{1}{3}\, a\, T^4$, where a is a constant.

a) Show that the polytropic index $n = 3$ for such a star. Obtain an expression for κ. You can assume a constant mean molecular mass. [8]

b) It is convenient to parametrise the density $\rho(r)$ by $\rho(r) = \rho_0\, \phi(r)^n$, where ϕ is a function normalised to unity at the centre (i.e. $\phi(r = 0) = 1$) and ρ_0 can then be interpreted as the central volume mass density. For a star in hydrostatic equilibrium, derive the equation for the structure of a polytrope:

$$\frac{1}{\epsilon^2}\, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\epsilon^2 \frac{\mathrm{d}\phi}{\mathrm{d}\epsilon} \right) = -\phi^n \ ,$$

where $\epsilon \equiv r/r_0$,

$$r_0 = \left[\frac{(n+1)\, \kappa\, \rho_0^{\frac{1-n}{n}}}{4\pi\, G} \right]^{\frac{1}{2}} \ ,$$

and G is the universal gravitation constant. [9]

c) What is the physical significance of the first zero of the function ϕ, i.e. ϵ_0 such that $\phi(\epsilon_0) = 0$? [2]

d) Show that the total mass of this polytrope is given by

$$M = -4\pi \left[\frac{(n+1)\, \kappa}{4\pi\, G} \right]^{\frac{3}{2}} \rho_0^{\frac{3-n}{2n}} \left(\epsilon^2\, \frac{\mathrm{d}\phi}{\mathrm{d}\epsilon} \right)_{\epsilon_0} \ ,$$

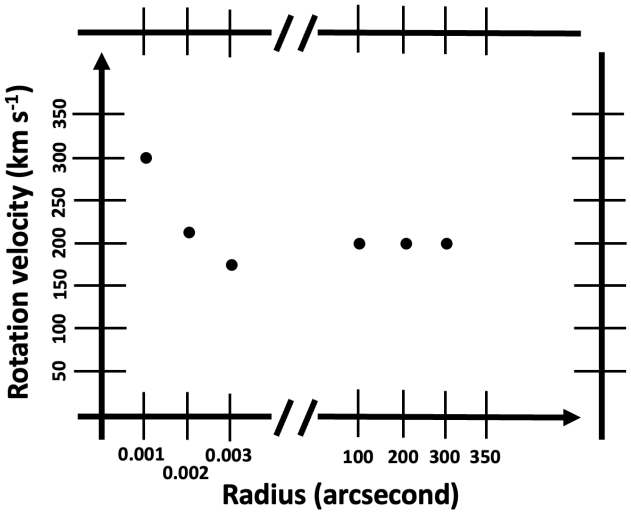
where the last term is evaluated at $\epsilon = \epsilon_0$. [6]

2. The observed Tully-Fisher relation of disk galaxies is

$$\mathrm{M}_V = -8.0\,[\log_{10}(V_{\rm rot}/\mathrm{km\ s^{-1}}) - 2.3\,] - 20.5,$$

where M_V is the total absolute magnitude of a galaxy in the V band and $V_{\rm rot}$ is the galaxy rotation velocity.

A galaxy with total apparent magnitude in the V band $m_V = 11.0$ has the measured rotation curve shown in the figure below. Note the break in radius along the abscissa.



A stellar population study reveals that the galaxy bulge contains only 10% of the total stellar mass, and that three Solar masses of stars are required to produce one Solar luminosity in the V band. You can ignore the effects of dust.

a) Calculate the distance to the galaxy and thus the true scale of the radii in the figure above. [6]

b) Estimate the total mass (in Solar masses) of the galaxy. State your assumptions, if any. [5]

c) Estimate the total stellar mass (in Solar masses) of the galaxy. Compare the total and stellar masses of the galaxy, and discuss whether the ratio is as expected given the mean composition of the universe. [7]

d) Show that the inner data points of the rotation curve are consistent with a spatially-unresolved central mass concentration. Estimate the mass (in Solar masses) of this spatially-unresolved central mass concentration. Discuss whether this mass is as expected given our knowledge of correlations between black hole and galaxy properties. [7]

[Assume that the absolute magnitude of the Sun in the V band is $\mathrm{M}_{V,\odot} = 4.8$.]

3. In cosmology, we can define conformal time η in term of the physical time t through $\mathrm{d}t = R\mathrm{d}\eta$, where R is the scale factor of the universe, that is a function of t or equivalently η.

a) Show that $\eta \propto R^{\frac{1}{2}}$ in a matter-dominated universe. [6]

b) Show that $\eta \propto R$ in a radiation-dominated universe. [4]

c) In a flat universe with only matter and radiation, with equality of matter and radiation energy density at $R = R_{\mathrm{eq}}$, show that

$$H^2 = H_0^2\,\Omega_{\mathrm{m},0}\left(\frac{1}{R^3} + \frac{R_{\mathrm{eq}}}{R^4}\right) ,$$

where H is the Hubble constant, H_0 is the Hubble constant today, and $\Omega_{\mathrm{m},0}$ is the matter density today expressed as a fraction of the critical energy density of the universe. [6]

d) In that universe, show that

$$\eta = \frac{2}{H_0\,\Omega_{\mathrm{m},0}^{\frac{1}{2}}}\left(\sqrt{R + R_{\mathrm{eq}}} - \sqrt{R_{\mathrm{eq}}}\right) .$$

[3]

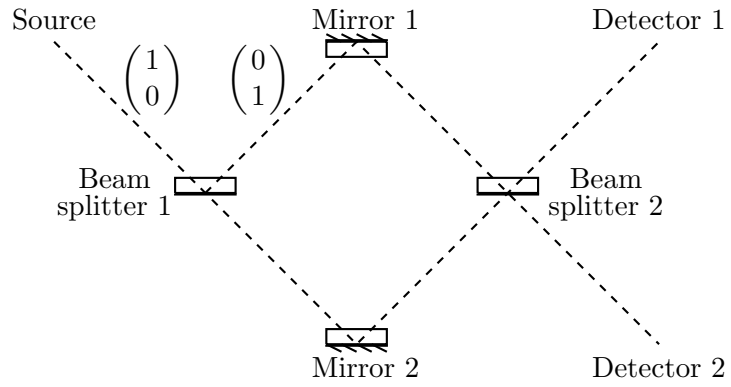
e) In that universe, what is the conformal time today? What was the conformal time at recombination? [6]

[Assume that the Hubble constant today $H_0 = 70$ km s^{-1} Mpc^{-1}.]

Section S03 QUANTUM IDEAS

1. (a) Describe briefly the wave and particle behaviours of photons in the double-slit experiment. Devise a method relying on this set-up to decide whether two glass plates have the same thickness or not. Discuss whether your method is operationally more efficient than the conventional approach of measuring the thickness of each piece and then comparing the results. [5]

(b) The set-up for a Mach–Zehnder interferometer is shown in the figure below, involving two 50-50 beam splitters and two perfect mirrors. If the source is turned on, what fraction of the original intensity arrives at each of the two detectors? Justify your answer. You can assume that the beam splitters are made of glass coated with a dielectric on the semi-reflecting side, both of negligible thickness. [3]



(c) If the vector $(1\ 0)^T$ describes a photon propagating at $-45°$ and the vector $(0\ 1)^T$ describes a photon propagating at $+45°$, as indicated in the figure above, show that the effect of each of the two beam splitters can be represented by the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Find the matrix describing the combined effect of the two mirrors. Verify that these matrices are unitary and explain why this is relevant. Is it also possible to describe the effect of the two detectors by matrices? [7]

(d) Introduce a glass plate of unknown thickness and refractive index n between the first beam splitter and mirror 1. Describe the evolution of the state of photons fired from the source and find the probability for photons to be detected by each of the two detectors. What information can you obtain about the thickness of the glass plate? [6]

(e) You can gain more information about the thickness of the glass plate by introducing a second glass plate next to the first one, capable of introducing a phase shift of $\pi/2$ (a quarter of a wavelength). Does this uniquely fix the thickness of the plate? [4]

2. (a) Explain the relevance of the Stern–Gerlach (SG) experiment for understanding the quantum properties of the electron. Why did the original SG experiment involve a beam of silver atoms and not a beam of electrons? [3]

In the rest of the question we will refer to electrons where silver atoms were actually used originally.

(b) Let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote, respectively, the spin-up and spin-down z-basis states of the electron, that is, the states in which the z-component of the electron spin is always measured to be $+\hbar/2$ and $-\hbar/2$, respectively. Is it possible to distinguish between the two x-basis spin states

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \ , \qquad |\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \tag{5}$$

using: (i) a vertical SG apparatus; (ii) a horizontal SG apparatus; (iii) a 45° diagonal SG apparatus? Explain your reasoning.

(c) Consider a general spin state

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \ , \qquad \alpha, \beta \in \mathbb{C} \ .$$

Calculate the probabilities for measuring spin-up and spin-down in the z-direction and explain what it means that $|\psi\rangle$ is a superposition of z-basis states. Is there a measurement context with respect to which $|\psi\rangle$ is not a superposition? If so, find this measurement context by assuming that α and β are real. [8]

(d) Explain how you can prepare a beam of electrons in the state $|\psi\rangle = |\uparrow\rangle$ using an SG apparatus. [2]

(e) Consider a beam of electrons prepared in the state $|\uparrow\rangle$. The beam passes through an SG apparatus aligned in the x-direction, splitting into two beams. The $-\hbar/2$ beam is filtered out and the $+\hbar/2$ beam is passed through a second SG apparatus aligned in the z-direction. What fraction of the original beam is measured in the second SG apparatus to have spin-down? Express this as a probability. [3]

(f) Consider again a beam of electrons prepared in the state $|\uparrow\rangle$. Pass it through N tilted SG devices followed by positive spin filters. The axes of the SG devices lie in the zx-plane, starting at an angle π/N with the z-axis and increasing in increments of π/N. (The system considered in part (e) corresponds to the case $N = 2$.) What is the probability to measure spin $|\downarrow\rangle$ in the final measurement along the z-axis, computed as in part (e) with respect to the intensity of the original beam? What happens when N is large (e.g., $N = 10$ or $N = 100$)? Compute this probability in the limit $N \to \infty$ (the quantum Zeno effect). [4]

Hint: The two states of definite spin for a measurement in the zx-plane at an angle θ with the z-axis are given by

$$|\psi_\theta^+\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle$$
$$|\psi_\theta^-\rangle = \sin(\theta/2)|\uparrow\rangle - \cos(\theta/2)|\downarrow\rangle \ .$$

3. (a) Define quantum entanglement and list two of its applications. [4]

Consider a two-qubit system described by the normalised wave function

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \ , \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C} \ .$$

(b) What is the probability of measuring the first qubit as 0? What is the (normalised) state of the system after measuring the first qubit as 0? What is the probability of measuring the first qubit as 1? What is the (normalised) state of the system after measuring the first qubit as 1? [4]

(c) Quantum entanglement is signalled by the presence of correlations between measurement probabilities on the two qubits. It can be shown that a necessary condition for the absence of such correlations is $|\alpha\delta| = |\beta\gamma|$. To show this, suppose that the system is prepared in the state $|\psi\rangle$ a large number N of times. The first qubit is measured first with an outcome of 0 obtained n times and an outcome of 1 obtained $m = N - n$ times. The second qubit is subsequently measured: out of the n times in which the outcome of the first measurement was 0, the second qubit is measured n_0 times as 0 and n_1 times as 1, with $n = n_0 + n_1$. Similarly, out of the m times in which the outcome of the first measurement was 1, the second qubit is measured m_0 times as 0 and m_1 times as 1, with $m = m_0 + m_1$. Compute the probabilities n_0/n and m_0/m. Explain why equating them is equivalent with the statement that the measurement probabilities on the two qubits are un-correlated and derive the condition $|\alpha\delta| = |\beta\gamma|$. [7]

(d) Show that a product state satisfies the stronger condition $\alpha\delta = \beta\gamma$ and that it can never exhibit correlations between measurement probabilities on the two qubits. [6]

(e) Suppose Alice has access to the first qubit of an entangled pair and Bob to the second. When Alice makes a measurement on her qubit, Bob's qubit instantaneously changes its state. Can Alice and Bob use entanglement to transmit information faster than the speed of light? Why or why not? [4]