A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 8 June 2023, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Using the transformation $u = x + y$, $v = y/(x + y)$, evaluate the integral

$$I = \iint \frac{\exp(x + y)}{x + y}\,\mathrm{d}x\,\mathrm{d}y$$

over the area enclosed by the lines $x = 0$, $y = 0$ and $x + y = 1$. [6]

2. A particle moving in the xy plane experiences a force $\mathbf{F} = (ax^2y, by^2x)$ where a and b are constants. Calculate the work done by the force in moving the particle from $(0, 0)$ to $(1, 1)$ along the curve $y = x^2$. Does the work done depend on the path chosen? Justify your answer. [5]

3. Evaluate $\int_S \nabla f \cdot \mathrm{d}\mathbf{S}$, where $f = (x + 1)^2 + y^2 + z^2$ and S is the closed surface formed by the hemisphere $x^2 + y^2 + z^2 = a^2, z > 0$ and the xy plane:

(i) directly as a surface integral, and [5]

(ii) by using the divergence theorem. [2]

4. i) Find the MacLaurin series for the function $x/\sin(x)$ up to terms of order x^4. [3]

ii) Evaluate the following limit:

$$\lim_{x\to\pi/4} \frac{1}{\pi - 4x} \int_{\frac{\pi}{4}}^{x} \frac{\sin^2 t}{t^2}\,\mathrm{d}t\,.$$

[3]

5. If $f \equiv f(x, y, z)$ and $g \equiv g(x, y, z)$, find the following partial derivatives in terms of partial derivatives of f and g with respect to x, y and z:

(i) $\left(\dfrac{\partial f}{\partial g}\right)_{x,y}$; [2]

(ii) $\left(\dfrac{\partial f}{\partial x}\right)_{z,g}$. [3]

6. The dispersion relation for gravity-driven waves on water is

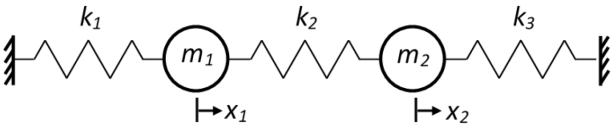
$$\omega^2 = gk\tanh(kh)\ ,$$

where ω is the angular frequency, k the wavenumber, g the acceleration due to gravity and h the water depth. Derive expressions for the phase velocity c_p and group velocity c_g as functions of k in each of the limits i) $kh \ll 1$ (depth $\ll$ wavelength) and ii) $kh \gg 1$ (depth $\gg$ wavelength). On the same axes sketch $c_p/\sqrt{gh}$ and $c_g/\sqrt{gh}$ as functions of (kh) for a fixed depth, indicating at least one characteristic value on each axis for scale. [6]

7. A ball bounces perfectly elastically on a horizontal surface reaching an identical maximum height h on each bounce. The time spent by the ball within a range of heights is proportional to the probability of finding it in that range. Find the probability distribution $P(x)$ such that $P(x)\,\mathrm{d}x$ is the probability that its height, sampled over an integral number of bounces, lies between x and $x + \mathrm{d}x$. [5]

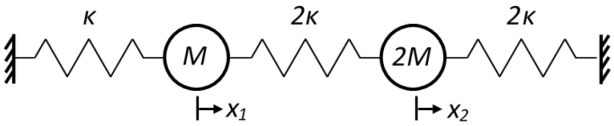
Section B

8. Two particles with masses m_1, m_2 are connected to fixed points and to each other by springs of negligible mass with spring constants k_1, k_2 and k_3 as shown. The particles are free to move in one dimension. Their displacements from their equilibrium positions are x_1, x_2 as shown.



(a) Write down equations of motion for the two particles. Explain what is meant by the phrase "normal mode". [4]

Now consider the special case $k_1 = \kappa$, $k_2 = k_3 = 2\kappa$, $m_1 = M$, $m_2 = 2M$.



(b) Show that the angular frequency of one normal mode is $\omega_a = \sqrt{\kappa/M}$ and find the frequency ω_b of the other normal mode. What are the relative displacements of the two particles in each normal mode? [10]

(c) Find x_1 and x_2 at times $t > 0$ for each of the following initial conditions:
(i) the masses are initially at rest with $x_1 = \alpha$, $x_2 = 0$;
(ii) the masses are initially in their equilibrium positions with velocities $\mathrm{d}x_1/\mathrm{d}t = 0$, $\mathrm{d}x_2/\mathrm{d}t = v$. [6]

9. a) Prove that

$$\nabla\times(\mathbf{a}\phi) = \phi\nabla\times\mathbf{a} - \mathbf{a}\times\nabla\phi \ ,$$

where $\mathbf{a}$ is a variable vector and ϕ is a variable scalar. [4]

b) i) By applying Stokes' theorem to $\mathbf{b}\phi$, where $\mathbf{b}$ is an arbitrary constant vector, show that

$$\oint_C \phi \, \mathrm{d}\mathbf{r} = -\int_S \nabla\phi \times \mathrm{d}\mathbf{S},$$

where C is the boundary of a surface S. [4]

ii) By explicitly evaluating both integrals, verify this result for the case in which $\phi = xy^2$ and S is the area in the plane $z = 0$ enclosed by the circle $x^2 + y^2 = c^2$. [12]

10. a) A transverse, sinusoidal wave of amplitude A and angular frequency ω propagates at speed c along a uniform string under tension T lying along the x axis. By considering the force exerted by the part of the string in the region $x < x_0$ on the part in region $x > x_0$, calculate the power $W(x,t)$ transmitted by the wave and show that its mean is $\frac{T}{2c}A^2\omega^2$. [7]

b) Two strings, each under tension T, lie along the x axis and are joined at $x = 0$. The string in the region $x < 0$ has constant density ρ_1 and the string in region $x > 0$ has density ρ_2.

i) If $y_1(x,t)$ and $y_2(x,t)$ are the displacements in $x < 0$ and $x > 0$ respectively, explain why the boundary conditions at $x = 0$ are

$$y_1(0,t) = y_2(0,t)$$

and

$$\frac{\partial y_1(0,t)}{\partial x} = \frac{\partial y_2(0,t)}{\partial x} \ ,$$

where $y_1(0,t) \equiv \lim_{x\to 0} y_1(x,t)$ etc. [2]

ii) A travelling wave $\mathrm{Re}\{\exp i(k_1 x - \omega t)\}$ is incident on the junction from the region $x < 0$. Given that wave speed is related to tension and density by $c^2 = T/\rho$, calculate the amplitudes and phases of the reflected and transmitted waves in terms of the densities ρ_1, ρ_2. Consider explicitly the cases $\rho_1 < \rho_2$ and $\rho_1 > \rho_2$. [11]

11. a) Write down the terms up to and including second order in the Taylor expansion of a function f of two independent variables about the point $(x,y) = (a,b)$. Use this expansion to derive conditions that identify a stationary point as a maximum, minimum or saddle point. [6]

b) Find the stationary points of the function $f(x,y) = xy\exp(x-y)$ and use the tests derived above to determine their type. [8]

Sketch f as a function of the parameter $s = (x-y)/\sqrt{2}$ along the line $x + y = 0$. [2]

Sketch a contour plot of the function indicating its stationary points, zeroes, regions in which the function is positive and negative and the directions in which it increases. [4]