A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 6th September 2023, 9:30 am – 12 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Show that

(i)
$$(-e)^{i\pi/2} = ie^{-\pi^2/2},$$

(ii)
$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

[5]

2. Sketch the loci in the complex plane representing the conditions $\mathrm{Re}(z^{-1}) = 1$ and $\mathrm{Im}(z^{-1}) = 1$. At what coordinates in the complex plane do these loci intersect? [5]

3. Solve the following differential equations

(i)
$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cos x,$$

[4]

(ii)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{1-x-y}.$$

[4]

4. A line is given by the equation

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})t$$

where t is a variable parameter, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the Cartesian axes x, y, z respectively, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The equation of the plane containing this line and the point (2,1,0) may be expressed in the form $\mathbf{r} \cdot \mathbf{a} = d$ where $\mathbf{a}$ is a unit vector and d is a constant. Find $\mathbf{a}$ and d, and explain their geometrical meaning. [5]

Find the volume of the tetrahedron with its four corners at: the origin, the point (2,1,0), and the points on the line with $t = 0$ and $t = 1$. [5]

5. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices satisfying the relation

$$\mathbf{A} = \mathbf{I} + \mathbf{B},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

(i) Write an expression for Trace($\mathbf{AB}$) in terms of b_{ij}, where b_{ij} are the matrix elements of $\mathbf{B}$. [3]

(ii) In the case that $\mathbf{B}^2 = 0$, show that $\mathbf{A}^n = \mathbf{I} + n\mathbf{B}$. [3]

6. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -2 \\ -4 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Hence solve the set of linear equations

$$\begin{aligned} x + y - 2z &= 5 \\ -4x + 2y + z &= -1 \\ x + y + z &= 2. \end{aligned}$$

[6]

Section B

7. Let

$$L = a_0 + \sum_{n=1}^{N} a_n \frac{\mathrm{d}^n}{\mathrm{d}t^n}$$

be a differential operator with constant coefficients.

(a) Apply L to the function e^{rt} to obtain the characteristic polynomial $p_L(r)$. Assume that $p_L(r) = 0$ has N distinct roots r_k. Show that the equation $L(y) = F_0 e^{\alpha t}$, where F_0 and α are constants and α is not a root of p_L, has the general solution

$$y(t) = y_0(t) + y_1(t),$$

where

$$y_0(t) = \sum_{k=1}^{N} c_k \, e^{r_k t}$$

is the solution to the homogenous equation $L(y) = 0$ (the c_k are constants), and

$$y_1(t) = \frac{F_0 \, e^{\alpha t}}{p_L(\alpha)}$$

is a particular solution to the inhomogenous equation $L(y) = F_0 \, e^{\alpha t}$. [6]

(b) Write the general solution the differential equation representing a damped, driven oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = F_0 e^{(-\beta + i\omega)t}$$

where $\gamma, \omega_0, F_0, \beta$, and ω are real constants, $\omega_0^2 > \gamma^2$, and $-\beta + i\omega$ is not a root of the equation's characteristic polynomial. Give an example of initial conditions which would uniquely determine the solution. [3]

(c) Calculate the magnitude and phase shift, relative to the driving term, of the particular solution $y_1(t)$ of (b). In what way does this particular solution fail if $-\beta + i\omega$ is, in fact, a root of the characteristic polynomial? [5]

(d) Find an expression for the magnitude and phase shift when $-\beta + i\omega$ is a root of the characteristic polynomial. What new behaviour is exhibited which was not present when $-\beta + i\omega$ was not a root? [6]

8. In a real vector space, the scalar product of a vector $\mathbf{v}$ with a vector $\mathbf{u}$ is written $\langle \mathbf{u}, \mathbf{v} \rangle$. It is required to satisfy the following four conditions

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \tag{C1}$$
$$\langle \mathbf{u}, \lambda\mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle \tag{C2}$$
$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \tag{C3}$$
$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \tag{C4}$$

where $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, λ is a scalar, and where in C4 there is equality if and only if $\mathbf{u} = \mathbf{0}$.

Consider the vector space $\mathcal{S}$ of the real symmetric $n \times n$ matrices.

(i) What is the dimensionality of $\mathcal{S}$? [1]

(ii) Propose a basis for $\mathcal{S}$ for the case when $n = 2$. [2]

(iii) Given that $\mathbf{A}$ and $\mathbf{B}$ are real symmetric $n \times n$ matrices, show that the association

$$\langle \mathbf{A}, \mathbf{B} \rangle = \text{Trace}(\mathbf{AB})$$

satisfies the stated conditions for a scalar product. [6]

(iv) Given that $\mathbf{B}$ is invertible, show that $\text{Trace}(\mathbf{B}^{-1}\mathbf{AB}) = \text{Trace}(\mathbf{A})$. [2]

(v) Prove that there are no matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\mathbf{AB} - \mathbf{BA} = \mathbf{I}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. [4]

(vi) Consider the alternative proposal for a scalar product in $\mathcal{S}$,

$$\langle \mathbf{A}, \mathbf{B} \rangle = \det(\mathbf{AB}).$$

Discuss the extent to which this alternative proposal satisfies the conditions C1 through C4 above. [5]

9. The Pauli matrices $\mathbf{S}_x$ and $\mathbf{S}_y$ are defined as

$$\mathbf{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) Find $\mathbf{S}_x^2$ and $\mathbf{S}_y^2$ and show that $\mathbf{S}_x$ and $\mathbf{S}_y$ do not commute. [3]

(b) The matrix $\mathbf{S}$ is now defined as

$$\mathbf{S} = \cos\theta\, \mathbf{S}_x + \sin\theta\, \mathbf{S}_y,$$

where θ is a real constant. Show that $\mathbf{S}^2 = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix. Find the elements of $\mathbf{S}$ and show that $\mathbf{S}^{-1} = (\mathbf{S}^T)^*$, where $\mathbf{S}^T$ is the transpose of $\mathbf{S}$ and * indicates the complex conjugate. [4]

(c) The matrix $\mathbf{S}$ and the vector $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfy the relation

$$\mathbf{S}\mathbf{u} = \lambda \mathbf{u},$$

where λ is a scalar. Show that a necessary condition for the above equation to have a non-trivial solution for $\mathbf{u}$ is $\det[\mathbf{S} - \lambda\mathbf{I}] = 0$.

Find the values of λ that yield non-trivial solutions for $\mathbf{u}$ and the corresponding vectors $\mathbf{u}$, assuming the normalisation $|x|^2 + |y|^2 = 1$. [9]

(d) The matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2 = \mathbf{A}$. Let $\mathbf{v} \neq \mathbf{0}$ be a vector satisfying the relation $\mathbf{A}\mathbf{v} = \mu\mathbf{v}$, where μ is a scalar. Find the possible values of μ. [4]

10. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors, and $\mathbf{c}$ is another vector such that

$$\mathbf{c} = x\mathbf{a} - y\mathbf{b},$$

where $x = \mathbf{b} \cdot \mathbf{b}$ and $y = \mathbf{a} \cdot \mathbf{b}$. Show by considering $\mathbf{c} \cdot \mathbf{c}$ that

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|.$$

[5]

(b) Show further that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. [4]

(c) The scalar product of two complex vectors $\mathbf{z}$ and $\mathbf{y}$, both with N components, is defined as

$$\mathbf{z} \cdot \mathbf{y} = \sum_{i=1}^{N} z_i\, y_i^*,$$

where z_i and y_i are the components of the two vectors and y_i^* is the complex conjugate of y_i.

Define a real vector $\mathbf{a}$ with $2N$ components consisting of the real and imaginary parts of the components of $\mathbf{z}$. The real vector $\mathbf{b}$ is similarly defined from the components of $\mathbf{y}$. Show that $\mathbf{a} \cdot \mathbf{b}$ is equal to the real part of $\mathbf{z} \cdot \mathbf{y}$. [5]

(d) Compute the cosine of the angle between the following pairs of vectors: (i) $(1, 0, i, i)$ and $(i, i, i, 0)$; (ii) $(e^{ix}, 1+i)$ and (i, e^{inx}), where x is a real constant, and n is an integer. [6]