A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 7 June 2023, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **NOT** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. The plane $H \in \mathbb{R}^3$ intersects the Cartesian coordinate axes at $(3,0,0), (0,-4,0)$ and $(0,0,6)$.

Find the unit normal to H and the parametric equation of the line L, perpendicular to H, that intersects the plane at the point $P(6,4,0)$. [5]

Express the equation of the line L in the form

$$\frac{(x-a)}{l} = \frac{(y-b)}{m} = \frac{(z-c)}{n}.$$

[1]

2. Express $\sin\left\{i\ln(iz \pm \sqrt{1-z^2})\right\}$ in its simplest form. [5]

3. (a) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, show that the components of $\mathbf{a}$ parallel and perpendicular to $\mathbf{b}$ are given by

$$\mathbf{a}_{\parallel} = \frac{(\mathbf{b}\cdot\mathbf{a})\mathbf{b}}{|\mathbf{b}|^2}; \qquad \mathbf{a}_{\perp} = \frac{(\mathbf{b}\times\mathbf{a})\times\mathbf{b}}{|\mathbf{b}|^2}.$$

[5]

(b) Prove Lagrange's identity,

$$|\mathbf{a}\times\mathbf{b}|^2 = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2.$$

[2]

4. $\mathbf{A}$ is a 2×2 matrix with elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Determine the characteristic polynomial $p_A(\lambda)$ of $\mathbf{A}$, expressing it in the form $\sum_{i=0}^{n} m_i\lambda^i$. Show by explicit evaluation that substituting $\mathbf{A}$ for λ in the characteristic polynomial yields a null result (Use $\mathbf{A}^0 = \mathbf{I}$). [3]

5. State the properties of an orthogonal matrix.

Matrix

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 3x & 2 & 2 \\ 2 & 1 & 3y \\ 2 & 3z & 1 \end{pmatrix}$$

is orthogonal. Find x, y and z. [5]

6. Show that W is a subspace of $V = \mathbb{R}^3$, where W consists of those vectors whose sum of components is zero, i.e. $W = \{(a, b, c) : a + b + c = 0\}$.

If U is the vector space of all $n \times n$ square matrices over a field F, show that Y is a subspace of U, where Y consists of all matrices which commute with a given matrix $\mathbf{T}$, i.e. $Y = \{\mathbf{A} \in U : \mathbf{AT} = \mathbf{TA}\}$. [5]

7. (i) Find the complete solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \cot x = 1,$$

subject to the initial condition $y = 2$ at $x = \pi/2$. [3]

(ii) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x + \sin(x).$$

[6]

Section B

8. A time-dependent displacement $x(t)$ satisfies the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\omega\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = F(t),$$

where $F(t)$ is a given function and λ, ω are positive constants.

(a) For the special case $F = 0$, obtain general solutions for $x(t)$ for the two cases $|\lambda| < 1$, $\lambda = 1$. [6]

(b) Obtain particular integrals for the cases (i) $F(t) = 1$, (ii) $F(t) = \cos\omega t$, and (iii) $F(t) = 1 + \cos\omega t$. What property of the differential equation accounts for the relationship between the first two particular integrals and the last? [8]

(c) For the case where $F(t) = \cos\omega t$ and $\lambda = 1$, obtain a solution such that x and $\mathrm{d}x/\mathrm{d}t$ vanish at $t = 0$. Show that in the resulting motion, at large t, x oscillates with amplitude $1/(2\omega^2)$. Does it lead or lag $F(t)$, and with what phase difference? [6]

9. (a) (i) Prove that the sum and products of the roots, x_i, of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ satisfy $\sum_{i=1}^{n} x_i = -a_{n-1}/a_n$ and, $\prod_{i=1}^{n} x_i = (-1)^n a_0/a_n$. [2]

Hence find the sum and products of the roots of $P = x^3 - 6x^2 + 11x - 6$. Show that $x = 1$ is a root. By writing $P = (x-1)Q$, where Q is a quadratic polynomial, find the other two roots. [3]

(ii) Show that the equation $(z+1)^n - e^{2\mathrm{i}n\theta}(z-1)^n = 0$ has roots $z = -\mathrm{i}\cot(\theta + m\pi/n)$, where m and n are integers. [3]

(iii) Using the results from (i) and (ii) above, show that

$$\prod_{r=1}^{n} \cot\left(\theta + \frac{m\pi}{n}\right) = \begin{cases} (-1)^{n/2} & \text{for } n \text{ even} \\ (-1)^{(n-1)/2}\cot n\theta & \text{for } n \text{ odd.} \end{cases}$$ [7]

(b) Draw the locus of the points $z = r\exp(\mathrm{i}\theta)$ with $r = |\sin(2\theta - \pi/3)|$ in an Argand diagram. For which angles θ is $|z|$ maximal? [5]

10. Matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of $\mathbf{A}$. [2]

(b) Find the eigenvalues of $\mathbf{A}$. [4]

(c) Find three normalized eigenvectors of $\mathbf{A}$, $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ spanning $\mathbb{R}^3$, such that $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \mathbf{v}_1 = 0$. Show explicitly that $\begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix}$ spans $\mathbb{R}^3$. [8]

(d) Construct a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with $d_{11} \leq d_{22} \leq d_{33}$. Evaluate $\mathbf{P}^{-1}$ and explicitly compute $\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{AP}$. [6]

11. (a) $\mathbf{A}$ is a square matrix. Show that $\mathbf{A} = \mathbf{B}+\mathbf{C}$, for some Hermitian matrix $\mathbf{B}$ and some skew-Hermitian matrix $\mathbf{C}$ (Matrix $\mathbf{C}$ is skew-Hermitian iff $\mathbf{C} = -\mathbf{C}^{\dagger}$). [3]

(b) Given $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -1 \\ 5 & 12 & -5 \end{pmatrix}$, find all vectors $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\mathbf{Au} = \mathbf{0}$. [4]

(c) Determine the values of k and β for which the following system of equations in unknowns x, y and z has:

(i) a unique solution

(ii) no solution

(iii) an infinite number of solutions. [5]

$$\begin{aligned} kx \;+\; y \;+\; z \;&= 1 \\ x \;+\; ky \;+\; z \;&= 1 \\ x \;+\; y \;+\; kz \;&= \beta \end{aligned}$$

Provide a geometrical interpretation for each combination of k and β above. [3]

(d) The Gram determinant of three real vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is defined as

$$G(\mathbf{a},\mathbf{b},\mathbf{c}) = \det \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}.$$

Show that

$$G(\mathbf{a},\mathbf{b},\mathbf{c}) = \left(\det \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \right)^2.$$

[3]

Further, show that if the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, their Gram determinant must be strictly positive. [2]