A10288W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP2: PHYSICS 2

Tuesday 6 June 2023, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of approved calculators is permitted.

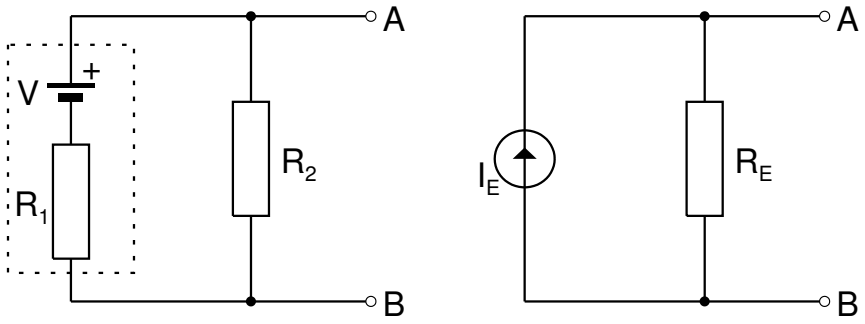
A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. The circuit on the left consists of a battery of constant voltage V and internal resistance R_1 in series with an external resistance R_2.

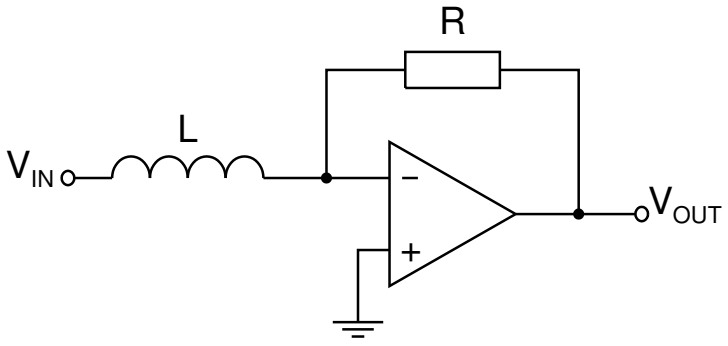


a) The circuit between terminals A and B can be simulated by the Norton equivalent circuit, on the right, consisting of a current I_E from an ideal current generator in parallel with a resistance R_E. Determine I_E and R_E. [2]

b) If $V = 10\,\text{V}$, $R_1 = 1\,\Omega$ and $R_2 = 2\,\Omega$, calculate (a proof is not required) the maximum power that could be dissipated in a resistor R_{AB} connected between A and B, and the value of R_{AB} required to achieve this. [3]

2. (a) State two characteristics of an 'ideal' op amp. [2]

(b) The following circuit contains an ideal op amp. Derive an expression for the output voltage V_{OUT} in terms of the input voltage V_{IN}.



[3]

3. (a) Show that the capacitance of two long coaxial conducting cylinders is given by

$$C = \frac{2\pi\epsilon_0 l}{\ln(b/a)}$$

where a and b are the inner and outer radii of the gap between the cylinders, and l is the length of the cylinders, with $l \gg b - a$. [3]

(b) Suppose that the cylinders are isolated, with fixed charges $+Q$ and $-Q$. Derive an expression for the force required to move the inner cylinder relative to the outer cylinder along their common axis. [3]

4. (a) Let P denote the path along a thin wire of arbitrary shape. Starting with an expression for the Lorentz force, obtain an integral expression for the induced EMF $V = \int_P \mathrm{d}V$ across the ends of a such a wire moving at velocity $\mathbf{v}$ in magnetic field $\mathbf{B}$. [2]

(b) An aeroplane with wingspan 10 m flies horizontally at $100\,\mathrm{m\,s^{-1}}$ through a region where the Earth's magnetic flux density (B) is $40\,\mu$T inclined at an angle 60° to the vertical. Calculate the EMF induced between the wing-tips of the aeroplane. [2]

5. (a) Show that a time-varying current $I(t)$ passing around a long solenoid generates an EMF $\mathcal{E}$ given by

$$\mathcal{E} = -\left(\frac{\mu_0 N^2 A}{l}\right)\frac{\mathrm{d}I}{\mathrm{d}t}$$

where l, A, N are the length, the cross-sectional area and the number of turns of wire of the solenoid. Assume that the solenoid is long compared to its diameter, so that the magnetic field outside the solenoid can be neglected, and that the resistance of the wire is also insignificant. [3]

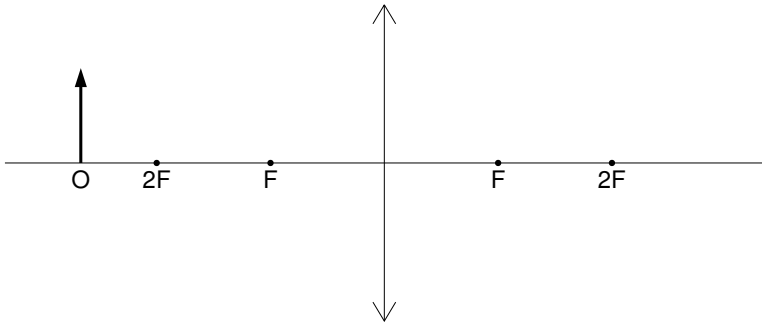
(b) Hence, derive an expression for the total energy stored in the magnetic field, and the energy density of the magnetic field within the solenoid as a function of B. [2]

6. Let U be the total electromagnetic energy U in a region R of free space. Let the vector $\mathbf{N}$ be the energy flux (power per unit area) in the electromagnetic field (this is called the Poynting vector). Conservation of energy implies

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\oint_{(R)} \mathbf{N}\cdot\mathrm{d}\mathbf{a}$$

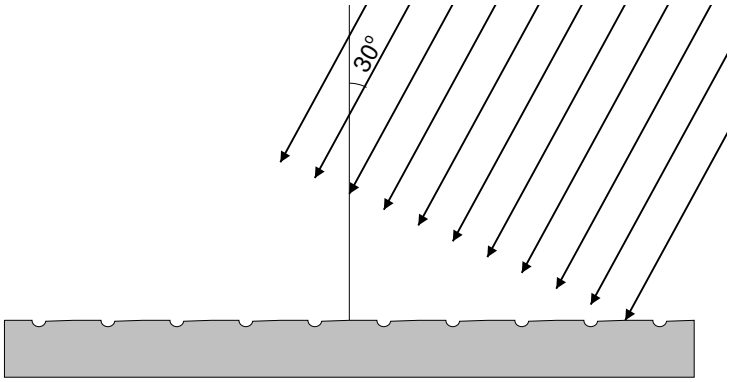
where (R) denotes the surface of the region and $\mathrm{d}\mathbf{a}$ is a surface element.
(i) Show that $\mathrm{d}u/\mathrm{d}t = -\nabla\cdot\mathbf{N}$ where u is the field energy per unit volume.
(ii) Starting with an expression for u in terms of the fields $\mathbf{E}$ and $\mathbf{B}$, take the time derivative and apply a vector identity to derive an expression for $\mathbf{N}$ in terms of the fields. [5]

7. a) The following sketch shows an object O placed more than twice the focal length F from a convex lens. Reproduce the sketch and use ray-tracing to construct the resulting image, showing its location, orientation and magnification. [2]



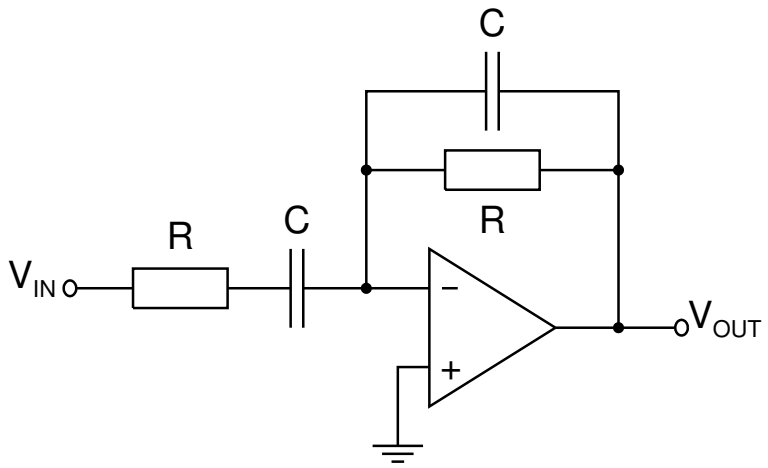
b) Determine the smallest possible distance between an object and a real image produced by a convex lens of focal length f. [3]

8. A collimated beam of light of wavelength 600 nm is incident on a reflective diffraction grating, in the plane perpendicular to the grooves, at 30° to the normal (as shown in the figure). If the grating has 250 lines/mm, calculate the angle of the diffraction maximum closest to the normal. [5]



Section B

9. The diagram shows a band-pass filter incorporating an 'ideal' op-amp.



a) The closed-loop complex gain is defined as

$$\tilde{G} = \frac{V_{\mathrm{OUT}}}{V_{\mathrm{IN}}}.$$

Show that, for a signal at frequency ω,

$$\tilde{G} = \frac{-\mathrm{i}\omega RC}{(1+\mathrm{i}\omega RC)^2}.$$

[5]

b) Derive an expression for the modulus of the gain, $|\tilde{G}|$. Determine its maximum value and show that this corresponds to the frequency at which the gain is real. [6]

c) Derive expressions for $\tilde{G}$ in the limits of high and low frequencies, and sketch the modulus and phase of $\tilde{G}$ as functions of frequency. [5]

d) Removing the component(s) which become redundant in the limit of high frequency, sketch a simplified version of this circuit which (approximately) reproduces its high-frequency behaviour, and verify this by determining the complex gain of the simplified circuit. Hence, or otherwise, suggest which resistance (input or feedback branch) should be altered, and whether increased or decreased, in order to *increase* the upper-frequency cut-off of the original filter. [4]

10. a) A flat annulus with inner radius a and outer radius b has uniform surface charge density σ. Derive an expression for the electrostatic potential V on the axis of the annulus at a distance h from its centre. [4]

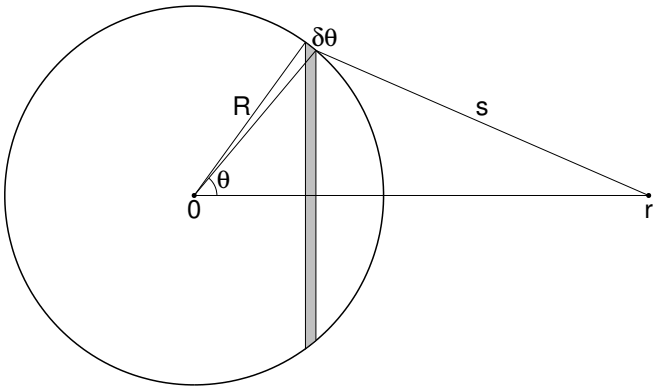
b) A thin-walled cylinder of radius a, length c also has uniform surface charge density σ. Derive an integral expression (you need not evaluate it) for the potential V on the cylinder's axis at a distance h beyond the end of the cylinder. [4]

c) Show that for $c \ll a$ the solution for the cylinder converges to

$$V = \frac{\sigma a c}{2\epsilon_0 s}$$

where $s = \sqrt{(a^2 + h^2)}$, and that this is also the solution for the annulus in part (a) for $c = b - a$ in the limit of small c. [2]

d) The diagram shows a spherical shell, radius R, with uniform surface charge density σ.



Show that the electrostatic potential at a point r due to a surface element subtending angle $\delta\theta$, shown by the shaded area in the diagram, is given by

$$\delta V = \frac{\sigma R}{2\epsilon_0 r}\delta s$$

where δs is the variation in s corresponding to $\delta\theta$. [Hint: differentiate the cosine rule to convert from θ to s.] [4]

e) Integrate over the surface of the sphere and express the potential $V(r)$ in terms of the total surface charge q to show that

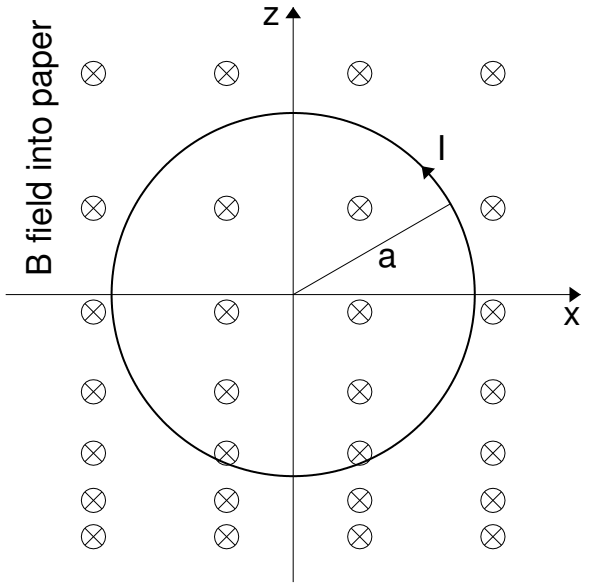
(i) for $r > R$ the potential is the same as if all the charge were concentrated at the origin, and

(ii) for $r < R$ the potential within the sphere is uniform, and determine its value. [6]

11. A magnetic field in the $+y$-direction decreases linearly with z as:

$$B_y(z) = B_0 - bz$$

where B_0 and b are positive constants.



(a) A circular loop of wire of radius a is placed normal to this field. A current I flows anticlockwise around the loop when viewed along the $+y$ direction, as shown in the diagram.

Write down an expression for the force d$\mathbf{F}$ on an element of wire d$\mathbf{s}$, and integrate this around the loop to show that the net force is in the $+z$ direction and given by

$$F_z = \pi a^2 I b,$$

and that this is also consistent with

$$\mathbf{F} = -\nabla U$$

where U represents the potential energy of the loop, treated as if it were a point-like magnetic dipole within the $\mathbf{B}$ field. [7]

b) The current source is switched off and the loop is allowed to drop under gravity in the $-z$ direction, maintaining the same orientation. Show that when the loop reaches a velocity v, the current induced in the loop is given by

$$I = \frac{\pi a^2 b v}{R}$$

where R is the electrical resistance of the loop. Hence show that at terminal velocity, the rate of loss of (gravitational) potential energy is balanced by the electrical energy dissipated in the loop (neglecting air resistance). [8]

c) Assuming that the material properties of the wire remain the same, calculate the new terminal velocity v' in terms of the original v in the case where

i) the thickness of the wire is halved,

ii) the radius a of the loop is doubled. [5]

12. a) State Huygens' Principle and explain how a long, narrow slit illuminated by a normally incident plane wave can be considered the source of a wave propagating as

$$\tilde{\Psi}(r,t) = \frac{\tilde{\Psi}_0(t)}{r}\exp(\mathrm{i}kr),$$

where r is the distance from the slit, k is the wavenumber, and $\tilde{\Psi}_0(t)$ represents the time-dependence of the incident wave at the slit. Why does the amplitude vary as $1/r$, and what other considerations become necessary for small r? [5]

b) A plane wave $A\exp[\mathrm{i}(kx-\omega t)]$ is incident upon a thin opaque plate located at $x=0$. The plate contains a narrow slit at $(x,y)=(0,Y)$, which can be considered infinitely long in the z-direction. Show that well beyond the slit the wave can be represented approximately, in polar coordinates, by

$$\tilde{\Psi}(r,\theta,t) \simeq \frac{\tilde{\Psi}_0(t)}{r}\exp(\mathrm{i}kr)\exp(-\mathrm{i}kY\sin\theta),$$

where r ($\gg Y$) is the distance from the origin, and θ is the angle from the x-axis. [4]

(c) A second, parallel slit is added at $y=-Y$. Show that the intensity of the resulting diffraction pattern at location (r,θ) is

$$\frac{\left|\tilde{\Psi}_0^2\right|}{r^2}\left(1+\cos 2\alpha\right),$$

where $\alpha = kY\sin\theta$. Hence show that for wavelength λ ($\ll Y$), the first minima occur at

$$\theta \simeq \pm\frac{\lambda}{4Y}.$$ [5]

(d) The two slits are replaced by a single slit of finite width extending from $-Y$ to $+Y$. Show that the resulting intensity pattern is a $(\mathrm{sinc})^2$ function and that the first minima occur at half the angle of the two-slit configuration in part (c). [6]