A14119W1

**FIRST PUBLIC EXAMINATION**

**Preliminary Examination in Physics**

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**SHORT OPTIONS**

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**TRINITY TERM 2022**

**Tuesday, 14 June**

**2:30 pm to 4:00 pm**

*Answer* **two** *questions from the option for which you have entered.*

*Start the answer to each question in a fresh book.*

*The use of approved calculators is permitted.*

*A list of physical constants and conversion factors accompanies this paper.*

*The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.*

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## Section S01 FUNCTIONS OF A COMPLEX VARIABLE

**1.** Let

$$F(z) = \exp\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right].$$

Classify the singularities of $F(z)$. [2]

The Laurent series of this function is

$$F(z) = \sum_{k=-\infty}^{\infty} F_k z^k.$$

Show that $F_k = F_{-k}$. [4]

Show that

$$F_k = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}\cos k\theta d\theta.$$

[6]

Show that

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{(z^2+1)^n}{z^{n+p+1}}dz = \begin{cases} \dfrac{n!}{(\frac{n-p}{2})!\,(\frac{n+p}{2})!} & \text{for } \frac{n+p}{2} \in \{0,1,2,\cdots\} \\ 0 & \text{otherwise} \end{cases}$$

[7]

Show that

$$F_k = \sum_{m=0}^{\infty}\frac{1}{2^{k+2m}m!(k+m)!}.$$

[6]

**2.** Show that a conformal transformation must be an analytic function. [4]

Consider a semi-infinite slab such that $x \geq 0$ and $0 \leq y \leq L$. Consider the mapping from $z = x + iy$ onto $w = u + iv$ such that

$$u = \cosh\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}y\right), \qquad v = \sinh\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right).$$

Show that this is a conformal transformation and find the form of $w(z)$. What region of the $w$-plane does the semi-infinite slab in $z$ map onto? [6]

Consider a real, harmonic function $V(x, y)$ defined on the semi-infinite slab which satisfies the boundary conditions:

$$\begin{aligned} V(x, 0) = V(x, L) &= V_0 \quad \text{for } x \geq 0 \\ V(0, y) &= -V_0 \quad \text{for } 0 < y < L. \end{aligned}$$

What is the corresponding boundary value problem in the $w$-plane? [2]

Show that

$$F(w) = \alpha \log(w + 1) + \beta \log(w - 1) + \gamma$$

is a harmonic function and use it find a solution to the boundary problem in the $w$-plane. [8]

Use the solution to find $V(x, y)$. [5]

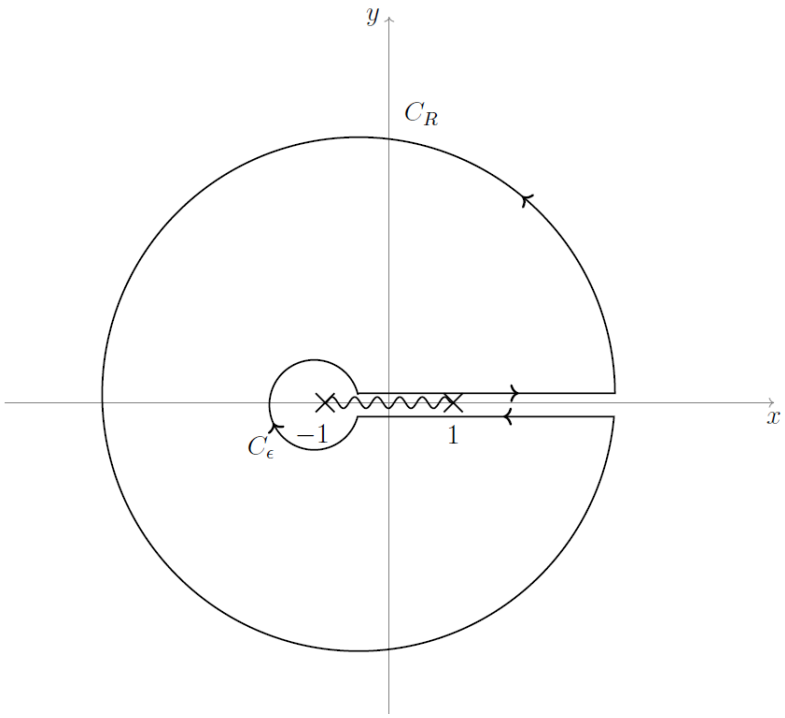
**3.** Consider the complex function

$$f(z) = \frac{1}{(1+z^4)\sqrt{1-z^2}}.$$

Find and characterize all the singularities of $f(z)$. [4]

If there are branch points, identify three different possible sets of branch cuts. [3]

Choose a branch cut on the x-axis that connects the two branch points you have found. Calculate the residues of all the singularities you have found. [9]



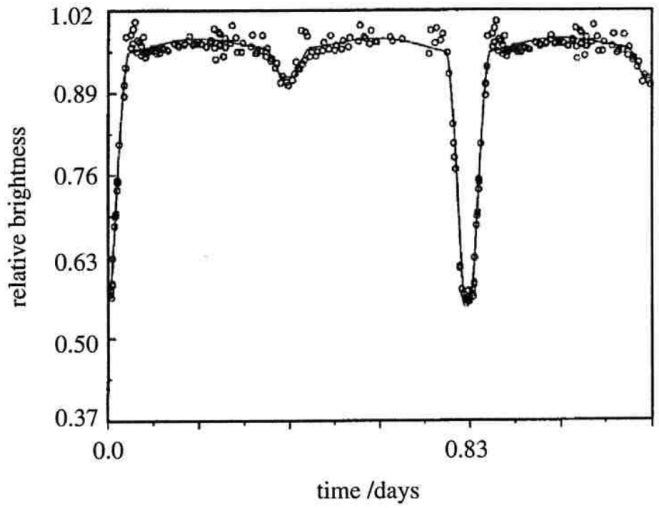

Consider the contour integral $\int_L f(z)dz$ where $L$ consists of the path illustrated in the figure. Taking the appropriate limits of the radii of $C_R$ and $C_\epsilon$, use this contour integral to calculate the value of

$$\int_{-1}^{1} dx \frac{1}{(1+x^4)\sqrt{1-x^2}}.$$

[9]

## Section S02 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

**1.** a) An eclipsing binary star system is viewed exactly edge-on and has the light curve shown in the figure below.




Briefly describe the two possible geometries corresponding to the primary and secondary dips in the light curve. Using the depths of the primary and secondary dips, estimate the two possible resulting sets of values of respectively the ratio of the radii and the ratio of the luminosities of the stars. [10]

b) Given a particular ratio of the stellar radii and of the stellar luminosities of the two stars, explain how one could in turn estimate the ratio of the surface temperatures of the two stars. Briefly describe two methods to more directly estimate the surface temperatures of the two stars. Assuming that the two stars are on the stellar main sequence, explain how one could estimate the ratio of their main sequence lifetimes. [8]

c) Briefly describe the main properties of stars on respectively the main sequence, red giant branch, horizontal branch, and white dwarf sequence. [7]

**2.** a) The linear momentum ($p$) flux across the layers of a star is given by $\frac{dp}{dt} = 4\pi R^2 P$, where $R$ is the stellar radius and $P$ the pressure. For radiation, $P = \frac{1}{3}aT^4$, where $a$ is constant and $T$ the temperature. Recalling that the energy of a photon is $E = cp$, where $c$ is the speed of light, and using the fact that radiation is absorbed (i.e. that the luminosity $L$ decreases with radius) as $\frac{dL}{dR} = -\kappa\rho L$, where $\kappa$ is the opacity and $\rho$ the matter volume density, derive the radiative transfer equation

$$L = -4\pi R^2 \, \frac{4ac}{3\kappa\rho} \, T^3 \, \frac{dT}{dR} \ .$$

[10]

b) In massive stars, the opacity is dominated by electron scattering ($\kappa = \kappa_{\rm th} =$ constant). Considering the radiative transfer equation and the equation of hydrostatic equilibrium, show that when radiation pressure dominates in the atmosphere of a massive star, the stellar mass-luminosity relation is given by $L \propto M$. Similarly, show that when gas pressure dominates ($P = \frac{\rho}{\mu m_{\rm H}} kT$, where $\mu$ is the mean molecular mass, $m_{\rm H}$ the mass of the Hydrogen atom, and $k$ Boltzman's constant), the stellar mass-luminosity relation is given by $L \propto M^3$. [9]

c) In low-mass stars, the opacity is given by $\kappa \propto \rho T^{-3.5}$ and gas pressure dominates in the atmosphere. Show that the stellar mass-luminosity relation is then given by $L \propto M^5$. [6]

**3.** a) Write the Friedmann equation that includes contributions from radiation (energy volume density $\rho_{\rm r}$), matter (energy volume density $\rho_{\rm m}$), a negative curvature ($-k$, where $k > 0$), and a cosmological constant ($\Lambda$). Considering this Friedmann equation and if necessary the fluid equation, describe the behaviour of each contribution as a function of the scale factor $R$. Which contributions will successively dominate as time passes? [8]

For the following sub-questions, assume that the universe is dominated by matter.

b) Estimate the age of the universe today. The distance light could have travelled in a time $t$ is known as the particle horizon, $r_{\rm p}(t) \equiv R(t) \int_0^t \frac{c\,dt'}{R(t')}$, where $c$ is the speed of light. What is the particle horizon today? [6]

c) The cosmic microwave background originates from the time of decoupling, when the scale factor of the universe was 1/1000 of its current value. What was the age of the universe then? What was the particle horizon then? How large would this particle horizon appear to be today? What angle would this particle horizon subtend on the sky today? [6]

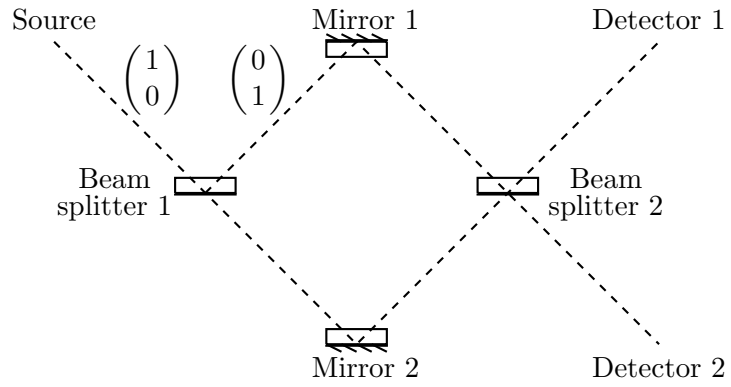
d) What is the significance of this angle? What does it imply about the very early universe? How can a large cosmological constant at very early times account for all of these facts? [5]

[ For radiation, the pressure $P$ is related to the energy volume density $\rho_{\rm r}$ as $P = \frac{1}{3}\rho_{\rm r} c^2$, where $c$ is the speed of light. The Hubble constant today is $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. ]

## Section S03 QUANTUM IDEAS

**1.** (a) Describe briefly the wave and particle behaviours of photons in the double-slit experiment. Devise a method relying on this set-up to decide whether two glass plates have the same thickness or not. Discuss whether your method is operationally more efficient than the conventional approach of measuring the thickness of each piece and then comparing the results. [5]

(b) The set-up for a Mach–Zehnder interferometer is shown in the figure below, involving two 50-50 beam splitters and two perfect mirrors. If the source is turned on, what fraction of the original intensity arrives at each of the two detectors? Justify your answer. You can assume that the beam splitters are made of glass coated with a dielectric on the semi-reflecting side, both of negligible thickness. [3]




(c) If the vector $(1\ 0)^T$ describes a photon propagating at $-45°$ and the vector $(0\ 1)^T$ describes a photon propagating at $+45°$, as indicated in the figure above, show that the effect of each of the two beam splitters can be represented by the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Find the matrix describing the combined effect of the two mirrors. Verify that these matrices are unitary and explain why this is relevant. Is it also possible to describe the effect of the two detectors by matrices? [7]

(d) Introduce a glass plate of unknown thickness and refractive index $n$ between the first beam splitter and mirror 1. Describe the evolution of the state of photons fired from the source and find the probability for photons to be detected by each of the two detectors. What information can you obtain about the thickness of the glass plate? [6]

(e) You can gain more information about the thickness of the glass plate by introducing a second glass plate next to the first one, capable of introducing a phase shift of $\pi/2$ (a quarter of a wavelength). Does this uniquely fix the thickness of the plate? [4]

**2.** (a) Explain the relevance of the Stern–Gerlach (SG) experiment for understanding the quantum properties of the electron. Why did the original SG experiment involve a beam of silver atoms and not a beam of electrons? [3]

In the rest of the question we will refer to electrons where silver atoms were actually used originally.

(b) Let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote, respectively, the spin-up and spin-down $z$-basis states of the electron, that is, the states in which the $z$-component of the electron spin is always measured to be $+\hbar/2$ and $-\hbar/2$, respectively. Is it possible to distinguish between the two $x$-basis spin states

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \ , \qquad |\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \tag{5}$$

using: (i) a vertical SG apparatus; (ii) a horizontal SG apparatus; (iii) a 45° diagonal SG apparatus? Explain your reasoning.

(c) Consider a general spin state

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \ , \qquad \alpha, \beta \in \mathbb{C} \ .$$

Calculate the probabilities for measuring spin-up and spin-down in the $z$-direction and explain what it means that $|\psi\rangle$ is a superposition of $z$-basis states. Is there a measurement context with respect to which $|\psi\rangle$ is not a superposition? If so, find this measurement context by assuming that $\alpha$ and $\beta$ are real. [8]

(d) Explain how you can prepare a beam of electrons in the state $|\psi\rangle = |\uparrow\rangle$ using an SG apparatus. [2]

(e) Consider a beam of electrons prepared in the state $|\uparrow\rangle$. The beam passes through an SG apparatus aligned in the $x$-direction, splitting into two beams. The $-\hbar/2$ beam is filtered out and the $+\hbar/2$ beam is passed through a second SG apparatus aligned in the $z$-direction. What fraction of the original beam is measured in the second SG apparatus to have spin-down? Express this as a probability. [3]

(f) Consider again a beam of electrons prepared in the state $|\uparrow\rangle$. Pass it through $N$ tilted SG devices followed by positive spin filters. The axes of the SG devices lie in the $zx$-plane, starting at an angle $\pi/N$ with the $z$-axis and increasing in increments of $\pi/N$. (The system considered in part (e) corresponds to the case $N = 2$.) What is the probability to measure spin $|\downarrow\rangle$ in the final measurement along the $z$-axis, computed as in part (e) with respect to the intensity of the original beam? What happens when $N$ is large (e.g., $N = 10$ or $N = 100$)? Compute this probability in the limit $N \to \infty$ (the quantum Zeno effect). [4]

*Hint: The two states of definite spin for a measurement in the $zx$-plane at an angle $\theta$ with the $z$-axis are given by*

$$|\psi_\theta^+\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle$$
$$|\psi_\theta^-\rangle = \sin(\theta/2)|\uparrow\rangle - \cos(\theta/2)|\downarrow\rangle \ .$$

**3.** (a) Define quantum entanglement and list two of its applications. [4]

Consider a two-qubit system described by the normalised wave function

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \ , \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C} \ .$$

(b) What is the probability of measuring the first qubit as 0? What is the (normalised) state of the system after measuring the first qubit as 0? What is the probability of measuring the first qubit as 1? What is the (normalised) state of the system after measuring the first qubit as 1? [4]

(c) Quantum entanglement is signalled by the presence of correlations between measurement probabilities on the two qubits. It can be shown that a necessary condition for the absence of such correlations is $|\alpha\delta| = |\beta\gamma|$. To show this, suppose that the system is prepared in the state $|\psi\rangle$ a large number $N$ of times. The first qubit is measured first with an outcome of 0 obtained $n$ times and an outcome of 1 obtained $m = N - n$ times. The second qubit is subsequently measured: out of the $n$ times in which the outcome of the first measurement was 0, the second qubit is measured $n_0$ times as 0 and $n_1$ times as 1, with $n = n_0 + n_1$. Similarly, out of the $m$ times in which the outcome of the first measurement was 1, the second qubit is measured $m_0$ times as 0 and $m_1$ times as 1, with $m = m_0 + m_1$. Compute the probabilities $n_0/n$ and $m_0/m$. Explain why equating them is equivalent with the statement that the measurement probabilities on the two qubits are un-correlated and derive the condition $|\alpha\delta| = |\beta\gamma|$. [7]

(d) Show that a product state satisfies the stronger condition $\alpha\delta = \beta\gamma$ and that it can never exhibit correlations between measurement probabilities on the two qubits. [6]

(e) Suppose Alice has access to the first qubit of an entangled pair and Bob to the second. When Alice makes a measurement on her qubit, Bob's qubit instantaneously changes its state. Can Alice and Bob use entanglement to transmit information faster than the speed of light? Why or why not? [4]