A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 8 September 2022, 9.30 am – 12.00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. x and y are defined by $x = \dfrac{at}{1-t^3}$, $y = \dfrac{at^2}{1-t^3}$, where t is a variable and a is a constant. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

2. Evaluate the limit $\lim_{x\to 0}\left[\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right]$ using i) l'Hôpital's rule and ii) a series method. [4]

3. If $u = u(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

[4]

4. State the divergence theorem. If $\mathbf{F} = \nabla\phi$, under what condition does

$$\iiint_V |\mathbf{F}|^2 \mathrm{d}V = \iint_S \phi\mathbf{F}\cdot\mathrm{d}\mathbf{S},$$

where V is the volume enclosed by S? [4]

5. Evaluate $\oint (2x - y + 4)\mathrm{d}x + (5y + 3x - 6)\mathrm{d}y$ anticlockwise around the triangle in the (x, y) plane with vertices $(0, 0)$, $(3, 0)$ and $(3, 2)$. [6]

6. (a) Find the volume of the region which is enclosed by the sphere $r = a$ and has angle $\theta \le \alpha$ from the z axis, where α is a constant $0 < \alpha < \pi$.

(b) Find the mass of this region if the density ρ is given by $\rho = \rho_0\,(1 - z/a)$. [7]

7. Make the change of variables $r = x + vt$ and $s = x - vt$ in the wave equation

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0$$

and hence obtain the solution to the wave equation in terms of r and s. In the case that F is a function of r only, and not s, how would the associated wave propagate? [6]

8. Consider the function $z(x, y) = x^2 + y^2 - 4x + y + xy + 3$.

(a) Show that z has only one stationary point and classify it.

(b) Calculate the gradient of z at the origin in the direction towards the stationary point. [5]

Section B

9. (a) Derive the vector triple product identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. [4]

(b) Consider a vector field $\mathbf{v}$ given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is a constant vector and $\mathbf{r}$ is the position vector.

(i) Sketch the field $\mathbf{v}$ in a plane perpendicular to $\boldsymbol{\omega}$. [2]

(ii) Prove that $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$. [4]

(iii) Imagine, for example, that $\mathbf{v}$ represents the velocity field of a moving fluid. What would happen to a matchstick dropped into the fluid at any point other than the origin? [2]

(c) Now consider the field $\mathbf{u} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$.

(i) Find constants a, b and c such that $\mathbf{u}$ is irrotational, i.e. $\nabla \times \mathbf{u} = 0$. [2]

(ii) Show that $\mathbf{u}$ can be expressed as the gradient of a scalar function ϕ and derive the exact form of ϕ as a function of x, y and z. [4]

(iii) Show that this particular ϕ is also a solution of Laplace's equation $\left(\nabla^2\phi = 0\right)$. What is the consequence for the divergence of the field $\mathbf{u}$? [2]

10. (a) Consider the surface S described by the function $z = f(x, y)$. Derive the following expression for the area of this surface:

$$\iint_S \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

[6]

(b) Evaluate the following two surface integrals:

(i)

$$\iint x^2 y^2 \mathrm{d}S$$

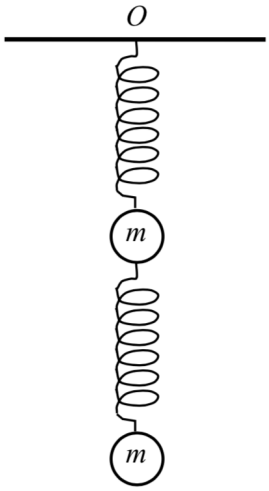
over the curved surface given by the part of the cone $x = \sqrt{y^2 + z^2}$ that lies between the planes $x = 1$ and $x = 2$; (Hint: try projecting onto the (y, z) plane.) [7]

(ii)

$$\iint \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

over the portion of the plane $y = 3 - x - 2z$ in the positive octant (i.e. $x, y, z \geq 0$), where $\mathbf{A} = x\mathbf{i} - x\mathbf{j} + z\mathbf{k}$. [7]

11. A mass m hangs below a fixed point O, to which it is connected by a spring of spring constant k. A second mass, of equal mass m, hangs below the first mass, to which it is connected by an identical spring, as shown in the diagram below.



(a) The two masses are set oscillating with small amplitudes in the vertical direction. Write down an equation of motion for each mass, assuming that the masses of the springs can be neglected. Explain why the gravitational force disappears from these equations if the position of each mass is defined relative to its equilibrium position. [4]

(b) Show that the angular frequency ω of each normal mode is given by an expression of the form

$$\omega^2 = a_i \frac{k}{m},$$

where a_i are constants to be determined. [6]

(c) Calculate the ratio of the amplitude of motion of the upper mass to that of the lower mass for both modes and sketch the resulting motion. [4]

(d) Suppose the fixed point O is replaced by a moving anchor with motion $z = z_0 \cos \omega_0 t$, where $\omega_0^2 = k/m$. Find the general solution for the motion of the two masses. [6]

12. A uniform string of mass per unit length ρ and tension T lies along the negative x-axis, and is terminated at $x = 0$ by connecting it to a damper able to move only in the y-direction. [In the following, you may assume $\partial y/\partial x$ is small, and that gravity can be neglected.]

(a) Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0\,,$$

describing small amplitude transverse waves on the string, and a formula for the wave speed c. [5]

(b) A transverse wave of form $y(x,t) = A\cos(\omega(t - x/c))$ is incident on the damper, which we idealise to have zero mass, and which provides a damping force with component in the y direction

$$F_y = -Z_d \frac{\partial y(x,t)}{\partial t}\bigg|_{x=0}$$

on the end of the string. In terms of A, Z_d, T, and c, find an expression for the amplitude of the reflected wave. [7]

(c) The total kinetic energy of the string is given by the integral

$$K = \frac{\rho}{2}\int_{-\infty}^{0} \left(\frac{\partial y(x,t)}{\partial t}\right)^2 \mathrm{d}x\,.$$

By considering the motion of an infinitesimal element of the string, derive, in terms of the displacement $y(x,t)$, a similar integral which gives the total potential energy U of the string. [3]

(d) For the incident wave in part (b), find an expression for the time-averaged power dissipated by the damper. [5]