A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 9 June 2022, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find the limit as i) $x \to 0$ and ii) $x \to \infty$ of the function

$$f(x) = \frac{1}{x} + \frac{\ln x}{\sin x - x}.$$

[4]

2. Find the derivative of $f(x) = \left(x^2 + 2\right)^{\frac{1}{x}}$. [3]

3. Consider the functions

$$f_1(x,y) = \frac{\sin y}{x} + e^x \cos^2 y, \qquad f_2(x,y) = x \sin y + e^x \cos 2y, \qquad f_3(x,y) = \ln x \cos y - e^x \sin 2y.$$

Two of these functions can be combined to form an exact differential $\mathrm{d}f$ of a function $f(x,y)$ as follows:

$$\mathrm{d}f = f_i(x,y)\mathrm{d}x + f_j(x,y)\mathrm{d}y.$$

Identify these two functions and determine the corresponding function f, explaining your reasoning. [5]

4. Determine the first order Taylor expansion of the function $f(x,y) = e^{(x^2+\sqrt{2}y)}$ about the point $(x_0, y_0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$. Does the error in this approximation to f grow fastest along the x or the y axis? [5]

5. For $\mathbf{F} = (-x, y, z)$, evaluate $\iint \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the surface of the plane $x + y + 2z = 4$ in the octant where x, y and z are all positive. [5]

6. Let $f_{X,Y}(x,y) = A \sin x \cos y$ define a joint probability density function normalised over the region bounded by the lines $y = \frac{\pi}{2} - x$, $y = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Find the value of the constant A and hence the expected value of the function $g(x) = \dfrac{1}{\sin x}$ over this region. [7]

7. A tube with an elliptical cross-section, defined by $x^2 + 4z^2 = 4$, is intersected by the planes $y + z = 1$ and $z = y + 1$. Find the volume of the enclosed region, by transforming to coordinates (r, θ, y) given by $x = 2r\cos\theta$ and $z = r\sin\theta$. [8]

8. An isotropic dispersion relation $\omega(k)$ has the property that $k^n\omega^2$ is constant. For what value of n will the energy associated with a wave packet propagate in the same direction as the individual peaks and troughs but with twice the speed? [3]

Section B

9. (a) Show that the identity

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = (\mathbf{r} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{r} + \mathbf{a}(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \mathbf{a})$$

holds in the specific case that $\mathbf{a}$ is a constant vector and $\mathbf{r}$ is the position vector. Is $\mathbf{a} \times \mathbf{r}$ a conservative field? [6]

(b) i) State Stokes' Theorem, and show that the line integral from A to B given by $\int_L P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ is independent of path if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. [4]

ii) Now consider the functions

$$P = yf(xy) - \frac{y}{x^2}\,g\left(\frac{y}{x}\right), \qquad Q = xf(xy) + \frac{1}{x}\,g\left(\frac{y}{x}\right),$$

defined within the region $x > 0$. For general functions f and g, show that a function $\phi(x,y)$ can be uniquely defined by the line integral

$$\phi(x,y) = \int_{(1,1)}^{(x,y)} P(x',y')\,\mathrm{d}x' + Q(x',y')\,\mathrm{d}y'.$$

Show that ϕ can be separated into integrals of f and g as follows

$$\phi(x,y) = \int_1^{xy} f(s)\,\mathrm{d}s + \int_1^{y/x} g(t)\,\mathrm{d}t,$$

and hence determine the difference in $\phi(x,y)$ between the points (a,b) and (ab,b^2), given that $f(s) = 1/s^2$. [10]

10. Consider the function $z(x,y) = xy\left(x^2-4\right)\left(y^2-4\right)$.

(a) Determine the stationary points of z. Sketch a contour plot of z in the region of the (x,y) plane where $x, y \geq 0$. Indicate on your sketch any maxima, minima, saddle points and zeros, and also the direction of the gradient of z. [9]

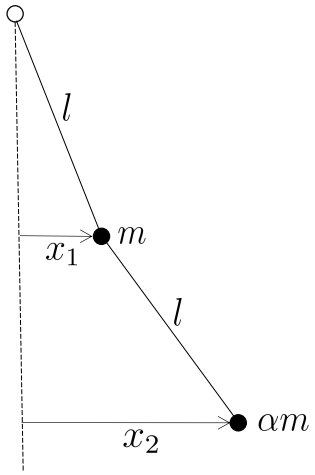
(b) For two-dimensional functions, the divergence theorem can be written as

$$\iint_A \nabla \cdot \mathbf{F}\,\mathrm{d}A = \oint_L \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}l = \oint_L M\,\mathrm{d}y - N\,\mathrm{d}x,$$

where L is a curve enclosing the area A, $\hat{\mathbf{n}}$ is the unit vector in the (x,y) plane perpendicular to L and $\mathbf{F} = (M,N)$. Verify this theorem by evaluating both the area and line integrals over the region where $0 \leq x \leq 2$ and $0 \leq y \leq 2$ in the case that $\mathbf{F} = \nabla z(x,y)$. [11]

11. Consider a double pendulum, consisting of two masses suspended by identical massless rods of length l. The upper and lower masses are m and αm respectively, where α is a constant. With x_1 and x_2 denoting the horizontal displacements of the upper and lower masses respectively, the equations of motion for small displacements can be written as

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -(1+\alpha)\frac{g}{l}x_1 + \alpha\frac{g}{l}(x_2 - x_1) \qquad \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -\alpha\frac{g}{l}(x_2 - x_1).$$



(a) Derive an expression for the frequencies of the normal modes of the system in terms of the constant α. [6]

(b) For the case where $\alpha = 1$ calculate the ratio of the amplitudes of the two displacements, and hence sketch the motion of the two masses. [6]

(c) Now assume that $\alpha \ll 1$ so that quadratic terms in α can be neglected. Determine the value that α should take for the lower frequency mode to have one quarter of the frequency of a single pendulum of length l and mass m. [4]

(d) Describe qualitatively what happens to the higher-frequency normal mode as $\alpha \to 0$. [4]

12. A uniform string of length L is fixed at all times at $x = 0$ and $x = L$. At time $t = 0$ the string has no displacement but is struck such that it acquires a transverse velocity $V(x)$, and thereafter it vibrates according to the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 y(x,t)}{\partial x^2},$$

where c^2 is constant.

(a) Show that this equation supports separable solutions and hence determine the frequencies of the normal modes. [5]

(b) Write down the general solution for the displacement of the string $y(x,t)$ and show that this can be written as a superposition of travelling waves. [4]

(c) Now consider the initial velocity to be given by

$$V(x) = V_0 \sin\frac{\pi x}{L}\left(1 + \cos\frac{\pi x}{L}\right) + \frac{2V_0}{3}\sin\frac{3\pi x}{L},$$

where V_0 is a constant. Which of the resulting modes will contribute most to the total energy of the motion? [2]

(d) Solve the system for the displacement $y(x,t)$ given this initial condition. [4]

(e) Determine the earliest time $t > 0$ that the string returns to being flat. Will the displacement of the string ever take the form of a pure sine wave? Explain your reasoning. [5]