A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 7 September 2022, 9.30 am – 12.00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is not permitted.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find a normalized vector that is normal to the plane passing through the points $\mathbf{s} = (3, 2, 4)$, $\mathbf{t} = (4, 2, 3)$ and $\mathbf{u} = (4, 4, 7)$. What is the distance between this plane and the origin? [5]

2. Find the rank of the matrix

$$\begin{pmatrix} 4 & 1 & 8 & 5 \\ 1 & 1 & 3 & 2 \\ 2 & -1 & 2 & 1 \\ 3 & 0 & \alpha & 3 \end{pmatrix} .$$ [5]

3. Complex number z has real and imaginary parts x and y respectively: $z = x + iy$ (x, y real). Sketch and describe the loci in the Argand diagram defined by the following equations and write down (or derive) the corresponding polynomial equations in x and y:

(i) $\arg\left(\dfrac{z-2}{z+i}\right) = 0$; [4]

(ii) $|z-4| + |z+4| = 10$. [5]

Hint: It is easier to consider geometrical interpretations of the equations than it is to resort to algebra.

4. Find a general solution to the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \cos x \ .$$ [5]

5. S and T are subsets of the vector space $V = \mathbb{R}^n$, the set of n-dimensional vectors $\mathbf{x}$. S comprises all eigenvectors of a particular matrix $\mathbf{M}$:

$$S = \{\mathbf{x}_i \in \mathbb{R}^n : \mathbf{M}\mathbf{x}_i = \lambda_i \mathbf{x}_i\}.$$

T comprises all vectors satisfying $\mathbf{M}\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a particular vector:

$$T = \{\mathbf{x}_j \in \mathbb{R}^n : \mathbf{M}\mathbf{x}_j = \mathbf{b}\}.$$

Under what conditions (if any) are S and T (separately) vector subspaces of V? Explain your reasoning. [6]

6. Find in the form $re^{i\theta}$ all values of complex number z which satisfy $z = (1+i)^{-\frac{1}{3}}$. Plot these solutions in an Argand diagram. [3]

7. Solve the following equation subject to the boundary condition $y(x{=}1) = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + 2y^3}{x(x^2 + 3y^2)}. \qquad [7]$$

Section B

8. (a) Describe the locus of vector $\mathbf{r}$ in each of the following cases, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a variable scalar:

$$\begin{aligned} &\text{(i)} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}; \\ &\text{(ii)} \quad (\mathbf{r} - \mathbf{a}).\mathbf{b} = 0; \\ &\text{(iii)} \quad (\mathbf{r} - \mathbf{a}).(\mathbf{r} - \mathbf{b}) = 0. \end{aligned}$$

[6]

(b) Three planes are defined by the following equations:

$$\begin{aligned} x + 2y + 3z &= 1; \\ 3x - y + 2z &= 3; \\ x + \alpha y + z &= \beta. \end{aligned}$$

(i) Find a vector formula for the line of intersection between the first two planes.

(ii) Find α_c, the value of α for which there is no unique point of intersection between the three planes. Describe qualitatively the geometrical relationship between the planes in the case $\alpha = \alpha_c$.

(iii) Find β_c, the value of β for which, when $\alpha = \alpha_c$, the three equations can be solved simultaneously. Describe qualitatively the geometrical relationship between the planes in the case $\alpha = \alpha_c$, $\beta = \beta_c$. [14]

9. (a) $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices that have the same eigenvalues λ_i but different eigenvectors. By considering the matrix transformations that diagonalize $\mathbf{A}$ and $\mathbf{B}$, or otherwise, find a transformation matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}^\dagger\mathbf{B}\mathbf{U}$ and prove that $\mathbf{U}$ is unitary. [6]

(b) Consider the following real, symmetric matrices:

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues of $\mathbf{C}$ and of $\mathbf{D}$. [4]

(ii) Find three normalized vectors that are simultaneously eigenvectors of both matrices. For each eigenvector, identify both corresponding eigenvalues $\lambda_{C,i}$ and $\lambda_{D,i}$. [8]

(iii) The coordinate system is rotated such that the non-degenerate eigenvector of **D** (i.e. the eigenvector corresponding to the non-degenerate eigenvalue) is parallel to the new z axis. Using the new coordinate system, describe the effect of applying the linear transformation **D** to all points on an arbitrary surface. [2]

10. The equation of motion for the displacement x of a body of mass m subject to a periodic driving force, a harmonic restoring force and frictional damping is

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega^2 x = A\cos\nu t\,.$$

a) Write down expressions for the three forces acting on the body. [2]

b) In the steady state, $x(t) = x_0\cos(\nu t+\phi)$. Show that the amplitude of oscillation x_0 is given by

$$x_0 = \frac{A}{[(\omega^2-\nu^2)^2+\alpha^2\nu^2]^{1/2}}\,.$$

[5]

c) Find the time-averaged rate at which energy is dissipated by friction and verify that this is equal to the time-averaged power input through the driving force. [10]

d) For the lightly damped case ($\alpha \ll \omega$), find approximate values of the frequencies $\nu \simeq \omega \pm \delta$ at which the amplitude is $1/\sqrt{2}$ of its maximum value at $\nu \simeq \omega$. [3]

11. a) Prove that

$$\sum_{n=0}^{\infty} e^{-nx}\cos(nx) = \frac{1-e^{-x}\cos x}{1-2e^{-x}\cos x+e^{-2x}}$$

for real $x > 0$. [6]

b) Plot in an Argand diagram the points corresponding to $ia-1$ and $ia+1$ where a is a particular positive real number. Show that

$$\frac{ia-1}{ia+1} = e^{2i\phi}$$

where $\tan\phi = \dfrac{1}{a}$ and $0 < \phi < \frac{\pi}{2}$. [5]

c) A complex number z is written in terms of w as $z = \left(e^{2iw}-1\right)/2i$ where w is real.

i) Evaluate $\mathrm{Im}\dfrac{1}{z}$. [5]

ii) Sketch the locus of z in an Argand diagram as w increases from 0 to π. [4]