A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 8 June 2022, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Vector $\mathbf{r}$ can be expressed in terms of basis vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (where $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) \neq 0$) in two ways:

$$\begin{aligned} \mathbf{r} &= \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}\,; \\ \mathbf{r} &= \mu_1\mathbf{b}\times\mathbf{c} + \mu_2\mathbf{c}\times\mathbf{a} + \mu_3\mathbf{a}\times\mathbf{b}\,. \end{aligned}$$

Find the coordinates λ_1 and μ_1 in terms of vector $\mathbf{r}$ and basis vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [3]

2. By making an appropriate substitution, solve for $y(x)$ the equation

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3}{x^3} - 2\frac{y}{x}.$$ [6]

3. Find all roots of $\cos(z) = i$ in the form $z = x + iy$ where x and y are real. [6]

4. A particle moving in one dimension is subject to gravitational and frictional forces:

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\gamma\frac{\mathrm{d}z}{\mathrm{d}t} - g\,.$$

The particle is projected from the origin with vertical velocity $\mathrm{d}z/\mathrm{d}t = v_0$ at time $t = 0$. Find the velocity of the particle at later times. [6]

5. Consider the following transformation matrices:

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix};\ \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};\ \mathbf{C} = \frac{1}{4}\begin{pmatrix} -2 & -\sqrt{6} & -\sqrt{6} \\ \sqrt{6} & 1 & -3 \\ \sqrt{6} & -3 & 1 \end{pmatrix}.$$

(i) What are the effects on a general vector $\mathbf{r}$ of multiplication by $\mathbf{A}$ and (separately) by $\mathbf{B}$?

(ii) An "improper rotation" comprises a rotation about some axis (a "proper rotation") then reflection in a plane perpendicular to that axis. Matrix $\mathbf{C}$ represents a general rotation operation. Without attempting to transform $\mathbf{C}$ to a new coordinate system, determine whether it represents a proper or improper rotation and the angle of rotation. Explain your reasoning. [8]

6. The charge Q on a capacitor in series with an inductor, a resistor and a voltage source obeys the differential equation

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = 0$$

where $R^2 < 4L/C$. Solve for $Q(t > 0)$ where, at $t = 0$, the charge on the capacitor is Q_o and the current $(\mathrm{d}Q/\mathrm{d}t)$ is zero. [5]

7. A vector addition operation $\oplus$ is defined for the set of n-dimensional vectors, $\mathbb{R}^n$, by

$$\mathbf{x} \oplus \mathbf{y} \equiv \mathbf{x} + \mathbf{y} - \mathbf{a}$$

where $\mathbf{a}$ is a fixed vector and the addition and subtraction operations on the right hand side are defined conventionally.

(i) Find the "null vector" $\mathbf{0}$ and $\mathbf{x}'$, the inverse of general vector $\mathbf{x}$, such that

$$\begin{aligned} \mathbf{x} \oplus \mathbf{0} &= \mathbf{x}, \\ \mathbf{x} \oplus \mathbf{x}' &= \mathbf{0}. \end{aligned}$$

(ii) Construct a corresponding scalar multiplication operation $\odot$ such that $\mathbb{R}^n$, together with operations $\oplus$ and $\odot$, forms a vector space over the field $\mathbb{R}$ and show that, with these definitions, scalar multiplication $\odot$ is distributive over vector addition $\oplus$. (It is not necessary to test all properties of the vector space.) [6]

Section B

8. (a) In each of the following cases describe the locus of points defined by the three-dimensional position vector $\mathbf{r}$ such that

$$\text{(i)} \quad |\mathbf{r} - \mathbf{a}| = c$$

$$\text{(ii)} \quad \mathbf{r} = c\,\frac{\mathbf{R} \times \mathbf{a}}{|\mathbf{R}||\mathbf{a}|}$$

where $\mathbf{R}$ is a variable vector (variable in both magnitude and direction), $\mathbf{a}$ is a constant vector and c is a constant scalar. [6]

(b) Lines L_1, L_2 are defined in vector form as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$ respectively where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors and λ, μ are variable scalar parameters.

(i) Write down the vector joining general points on L_1 and L_2. [1]

(ii) Show that the shortest distance between these lines is given by $(\mathbf{a} - \mathbf{c})\mathbf{.n}$ where

$$\mathbf{n} = \frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}\,. \qquad [2]$$

iii) Find λ as a function of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ at the point at which L_1 is closest to L_2. [6]

(c) Points on two lines satisfy the equations

$$(x-3) = \frac{(4-y)}{2} = \frac{(z-1)}{3}$$

and

$$\frac{(x-2)}{\alpha} = (y+1) = \frac{(z-5)}{2}$$

where α is an undetermined constant. What is the value of α for which the lines intersect? [5]

9. (a) Find, in the form $r\exp(i\theta)$, all values of complex number z which satisfy the following equations. Plot these solutions in the Argand diagram.

$$\text{(i)} \quad z^5 = 1 + \sqrt{3}i$$
$$\text{(ii)} \quad z = \left(\sqrt{2}(1+i)\right)^i$$
$$\text{(iii)} \quad \mathrm{Re}\left\{\frac{z-1}{z+1}\right\} = 0$$

[12]

(b) By considering the binomial expansion of $(1+e^{i\theta})^N$, or otherwise, show that

$$\sum_{n=0}^{N} \binom{N}{n} \sin(n\theta + \phi) = 2^N \cos^N\left(\frac{\theta}{2}\right) \sin\left(\frac{N\theta}{2} + \phi\right)$$

where $\binom{N}{n}$ represents the binomial coefficient $\left(\dfrac{N!}{(N-n)!n!}\right)$. [8]

10. (a) Matrices $\mathbf{U}$, $\mathbf{V}$ are unitary. Show that their product $\mathbf{UV}$ is also unitary. [2]

(b) Matrix $\mathbf{M}$ and its adjoint $\mathbf{M}^\dagger$ have the same set of eigenvalues λ_i but different eigenvectors ($\mathbf{u}_i$, $\mathbf{v}_i$ respectively):

$$\begin{aligned} \mathbf{M}\mathbf{u}_i &= \lambda_i \mathbf{u}_i\,; \\ \mathbf{M}^\dagger \mathbf{v}_i &= \lambda_i \mathbf{v}_i\,. \end{aligned}$$

Show that $\mathbf{v}_i^\dagger \mathbf{u}_j = 0$ for $\lambda_i^* \neq \lambda_j$. [4]

(c) A matrix $\mathbf{A}$ can be transformed from one orthonormal basis to another by the unitary transformation $\mathbf{A'} = \mathbf{U}^\dagger\mathbf{A}\mathbf{U}$. State the corresponding transformation for column vector $\mathbf{x}$. For the specific examples

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & i \\ 0 & 3 & 0 \\ -i & 0 & 0 \end{pmatrix}\,,\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\,,$$

construct a symmetric, unitary matrix $\mathbf{U}$ that can be used to transform $\mathbf{A}$ to a basis in which it is diagonal and find transformed matrix $\mathbf{A'}$ and vector $\mathbf{x'}$ in this basis. [14]

11. a) Find the general solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{\sqrt{1+x^2}}\,.$$

[3]

b) Solve the equation

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^3$$

subject to the boundary condition $y(1) = 1$. [5]

c) Find the general solution of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + (\beta^2+1)y = e^x(1+\cos x)$$

where β is a real constant:

i) for $\beta^2 \neq 1$;

ii) for $\beta^2 = 1$. [12]