A10288W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP2: PHYSICS 2

Tuesday 7 June 2022, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

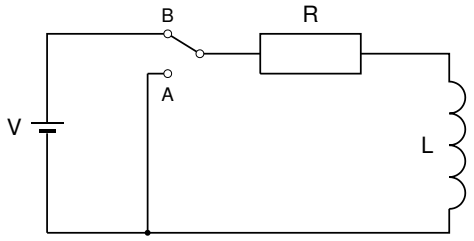
A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (i) At time $t = 0$ the circuit below has a current I_0 flowing clockwise ($I_0 \leq V/R$), and the switch is moved instantaneously from position B to position A. Derive an expression for the subsequent current $I(t > 0)$. [2]



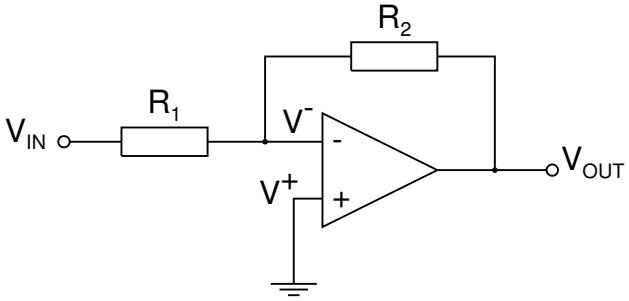
(ii) After a time Δt, the switch is moved from A back to B. Derive the current I_2 flowing after a time $2\Delta t$ in terms of I_0.

Hence, assuming that the switch is flipped continuously between A and B with constant time interval Δt, show that after a large number of cycles the current flowing at the time of the switch from B to A converges to:

$$I_{2n} = \frac{V}{R}\left(1 + \exp\left(\frac{-R\Delta t}{L}\right)\right)^{-1}.$$ [4]

2. (i) Apart from infinite gain, list two other assumptions associated with an 'ideal' op amp. [2]

(ii) The circuit below contains an op amp which can be considered 'ideal' except for a finite gain, A, where $V_{\mathrm{OUT}} = A(V^+ - V^-)$.



Show that the overall gain for this circuit is of the form

$$\frac{V_{\mathrm{OUT}}}{V_{\mathrm{IN}}} = -\frac{R_2}{R_1 + B}$$

and derive an expression for B. [2]

3. A current I flows along a long cylindrical conductor of radius a. Calculate the magnitude of the magnetic flux density B as a function of radius r within the cylinder for the cases where

(i) the current density is uniform throughout the cylinder, and [3]

(ii) the current density is zero at the centre, and increases linearly with radius. [3]

4. Point charges q_1 and q_2 are placed at polar coordinates $(r_1, 0)$ and $(r_2, 0)$ respectively. Write down an expression for the electrostatic potential at a point (r, θ).

Assuming that the line of zero potential can be represented by a circle centred on the origin, show that q_2 must equal $-q_1\sqrt{r_2/r_1}$, and determine the radius of the circle. [6]

5. (i) Write down Maxwell's equations for fields in a vacuum, with no electric charges or currents present, and show that these support plane wave solutions of the form

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \exp\left(i(\omega t - kx)\right) \\ \mathbf{B} &= \mathbf{B}_0 \exp\left(i(\omega t - kx)\right) \end{aligned}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant vectors, and derive an expression for the propagation speed c. [4]

(ii) If $\mathbf{E}_0 \equiv (0, E_0, 0)$ in Cartesian coordinates, derive the corresponding form of $\mathbf{B}_0$ and show that

$$\frac{|\mathbf{E}_0|}{|\mathbf{B}_0|} = c.$$ [2]

6. (i) A ray of light approaches from infinity along a path defined, in polar coordinates, by

$$r\cos\left(\theta - \theta_a\right) = d_a$$

where θ_a, d_a are constants ($\theta_a = 0$ may be assumed).

A sphere of radius r_b ($> d_a$) and refractive index n_b is centred on the origin. If the refractive index outside the sphere is n_a ($< n_b$) show that, on reaching the sphere, the ray is deflected to a new path defined by

$$r\cos\left(\theta - \theta_b\right) = \frac{n_a}{n_b} d_a$$

where θ_b is a constant which need not be determined. [3]

(ii) At the Earth's surface a star appears on the horizon. Calculate the minimum altitude above the surface that the same light ray would have if the Earth had no atmosphere. You can assume that the Earth is a sphere of radius 6.4×10^3 km and that the refractive index of air at the surface is 1.0003. Explain why the variation in refractive index with altitude does not affect the answer. [3]

7. (i) A slit of width w (and infinite height) is illuminated at normal incidence by a collimated beam of coherent light of wavelength λ. Derive the angular intensity distribution in the diffraction pattern at a distance much larger than w. [4]

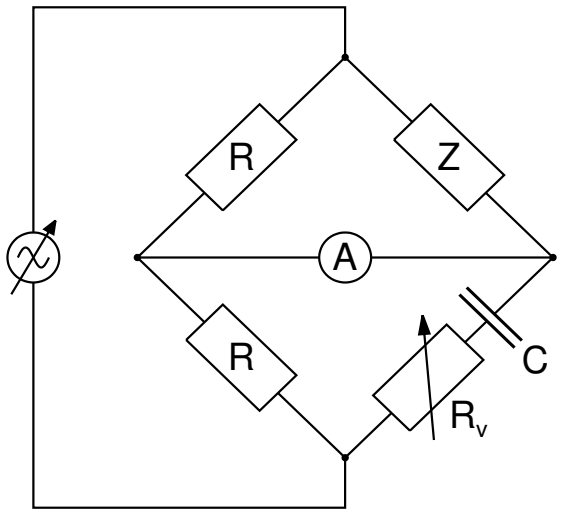
(ii) A satellite orbiting at an altitude of 840 km measures infrared emission from the Earth's surface at wavelength of 12 μm. If the instrument is to resolve surface features of size 100 m, what is the minimum width required for the collecting mirror? [2]

Section B

8. (a) A complex impedance has the form $Z = A + i\omega B$. Assuming that this can be matched to the impedance of a resistance R and capacitance C in series, determine the specific values of ω and R for this to occur. [3]

Determine whether or not impedance matching is possible if Z represents a resistance and an inductance (i) in series, and (ii) in parallel. [5]

(b) The bridge circuit below is used to determine the complex impedance Z, consisting of a resistance R_x in parallel with capacitance C_x, by varying the angular frequency ω of the voltage source, on the left, and adjusting the variable resistance R_v until the current amplitude through the ammeter A (assumed to have negligible impedance) is zero.



Show that, at balance, R_v is given by

$$\frac{C_x}{C} = 1 - \frac{R_v}{R_x}$$

and determine the value of ω. [8]

Assuming R_v is set for balance at some frequency ω_0, calculate the voltage at the ammeter in terms of the driving amplitude V_0 for (i) $\omega = 0$, (ii) $\omega = \omega_0$ and (iii) as $\omega \to \infty$. [4]

9. (a) A point charge $+q$ is placed at height h above an infinite conducting grounded plane. Show that the z-component of the electric field just above the plane is given by

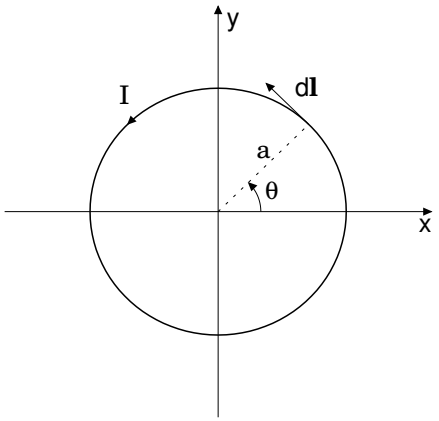
$$E_z = -\frac{q}{2\pi\epsilon_0}\left(\frac{h}{(a^2+h^2)^{\frac{3}{2}}}\right)$$

where a is the distance along the plane from the point directly underneath the charge. [6]

(b) A negative charge $-q$ is added a small distance d above $+q$ ($d \ll h$). By considering $\mathrm{d}E_z/\mathrm{d}h$, or otherwise, derive the modified expression for E_z. What is the significance of the circle of radius $a = h\sqrt{2}$ in the plane? [10]

(c) Calculate, for both the monopole and dipole configurations, the induced surface charge density on the plane directly underneath the point charges. [4]

10. (a) A circular loop of wire of radius a lies in the x-y plane centred at the origin with a current I flowing in the anticlockwise sense, as shown in the diagram. The loop lies within a uniform magnetic flux density $\mathbf{B} = (B_x, B_y, B_z)$. Derive an expression for the force d$\mathbf{F}$ acting on a small element of the loop, d$\mathbf{l}$, at angle θ from the x-axis, and show that the net force on the loop is zero. [5]



(b) Assuming that the loop is free to rotate about the x axis, derive an expression for the torque on the loop element, hence show that the net torque $\boldsymbol{\tau}$ on the loop about the x-axis is given by

$$\boldsymbol{\tau} = -I\pi a^2 B_y \hat{\mathbf{x}}$$

and show that this is consistent with the x component of the torque evaluated as

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$$

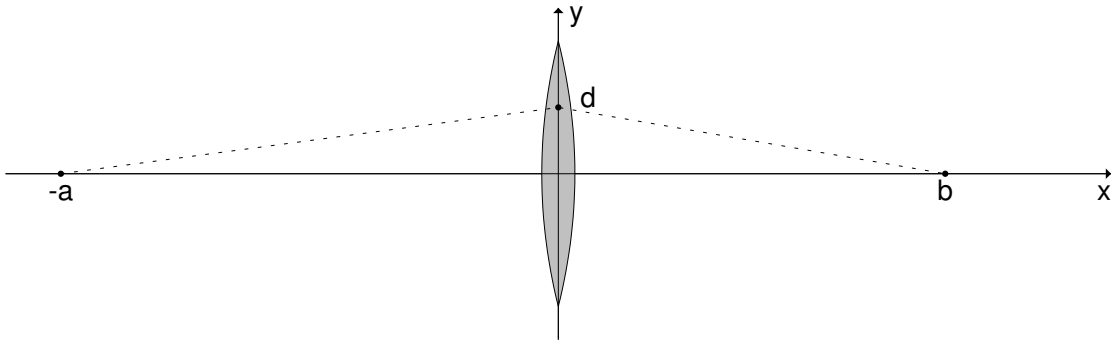
where $\mathbf{m}$ is the magnetic dipole moment of the loop. Why does $\mathbf{m} \times \mathbf{B}$ also have a y component? [10]

(c) The current is kept constant while the loop rotates. Show that the potential energy of the loop as a function of the rotation angle α can be expressed as

$$E = E_0 \cos(\alpha - \alpha_0)$$

and derive expressions for E_0 and α_0. [5]

11. (a) The diagram shows a thin lens at the origin with its optical axis along the x axis. A light ray that crosses the x axis at $x = -a$ passes through the lens at $y = d$ and is refracted so as to cross the x axis again at $x = b$. The refractive index of the lens is n and its thickness is t, which varies as $t = t_0 - \alpha y^2$, where α is a constant defining the curvature. The whole system is in a vacuum.



Assuming that $d \ll a, b$ and that t is sufficiently small that the paraxial approximation is valid, show that the time taken for a ray to travel from $x = -a$ to $x = b$ is

$$t = \frac{1}{c}\left[a\left(1 + \frac{d^2}{2a^2}\right) + b\left(1 + \frac{d^2}{2b^2}\right) + (n-1)(t_0 - \alpha d^2)\right]. \qquad [8]$$

(b) State Fermat's principle in optics. Hence, show that the focal length of the lens is given by

$$f = \frac{1}{2\alpha(n-1)}. \qquad [5]$$

(c) How does the expression for focal length change if the whole system is immersed in a medium of refractive index n'? Discuss the behaviour as n' is increased from 1 to n. [4]

(d) Suppose a medium of refractive index n has a vacuum-filled void in the shape of the convex lens. Determine whether or not the resulting focal length is equivalent to replacing the convex lens in part (a) with a concave lens of the same curvature $|\alpha|$ and refractive index. [3]