A14119W1

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SHORT OPTIONS

TRINITY TERM 2021

Tuesday, 15 June

Opening time 9.30 am UK time

You have one hour and thirty minutes to complete the paper and up to 30 minutes technical time to upload your answer file.

Answer **two** *questions.*

Start the answer to each question on a new page.

Write your candidate number, paper name and short option taken, paper code and a list of the questions that you have answered on the first page, and submit all of your answers as a single PDF file.

The use of approved calculators is permitted.

A list of physical constants and conversion factors will be available within Inspera during the examination

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

INDEX

Section	Subject	Page
S01	Functions of a Complex Variable	2
S02	Astrophysics: from Planets to the Cosmos	4
S03	Quantum ideas	7

Section S01 FUNCTIONS OF A COMPLEX VARIABLE

1. (a) Let

$$f(z) = \frac{1}{2+z} + \frac{\sin z}{z^2} - \frac{1}{z}$$

Determine and classify all the singularities of $f(z)$. [6]

(b) Expand $f(z)$ in a Laurent series in powers of z in the region $0 < |z| < 2$. [7]

(c) Calculate

$$I = \oint_\Gamma \frac{z}{z^4 - 10z^2 + 1} \mathrm{d}z$$

where Γ is a circle centred at $z = 0$ with radius 1. [6]

(d) Use your result for I to find

$$\int_0^{2\pi} \frac{1}{2 + \sin^2\theta} \mathrm{d}\theta$$

[6]

2. (a) State Cauchy's Residue Theorem. [3]

(b) Let us define $f(z)$ such that

$$f(z) = \frac{z^2}{z^4 + 1}.$$

Use Cauchy's Residue Theorem to calculate

$$\oint_{\Gamma_R} f(z)\mathrm{d}z,$$

where $\Gamma_R = \gamma_R \cup [-R, R]$ is a positively oriented closed contour and γ_R is a semi-circle with radius $R > 1$ and $\mathrm{Im}(z) \geq 0$. [8]

(c) Show that

$$\left| \oint_{\gamma_R} f(z)\mathrm{d}z \right| \leq \frac{\pi R^3}{|R^4 - 1|}.$$

[9]

Use the previous results to calculate

$$\int_0^\infty \frac{x^2}{x^4 + 1} \mathrm{d}x$$

[5]

3. (a) Consider the integral

$$J = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{e^{-\sqrt{z}+z}}{z} \mathrm{d}z,$$

where γ is a positive real number. Explain why $z = 0$ is a singularity but not a pole of the integrand. Explain why the halfline along the negative x-axis is a branch cut of the integrand. [4]

(b) Consider the closed contour, C in the figure, made up of six segments, C_1 to C_6; note that segments C_2 and C_6 have a radius R while the segment C_4 has radius ϵ. Show that

$$\oint_C \frac{e^{-\sqrt{z}+z}}{z} \mathrm{d}z = 0$$

[5]

(c) Show that

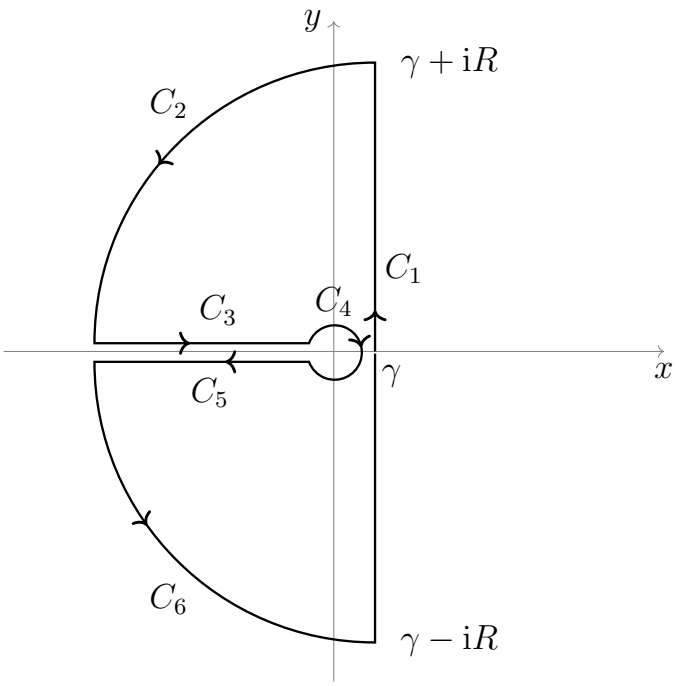
$$\int_{C_3} \frac{e^{-\sqrt{z}+z}}{z} \mathrm{d}z + \int_{C_5} \frac{e^{-\sqrt{z}+z}}{z} \mathrm{d}z = 2\mathrm{i} \int_0^\infty e^{-x} \sin(\sqrt{x}) \frac{\mathrm{d}x}{x}$$

[8]

(d) Show that

$$J = 1 - \frac{1}{\pi} \int_0^\infty e^{-x} \sin(\sqrt{x}) \frac{\mathrm{d}x}{x}$$

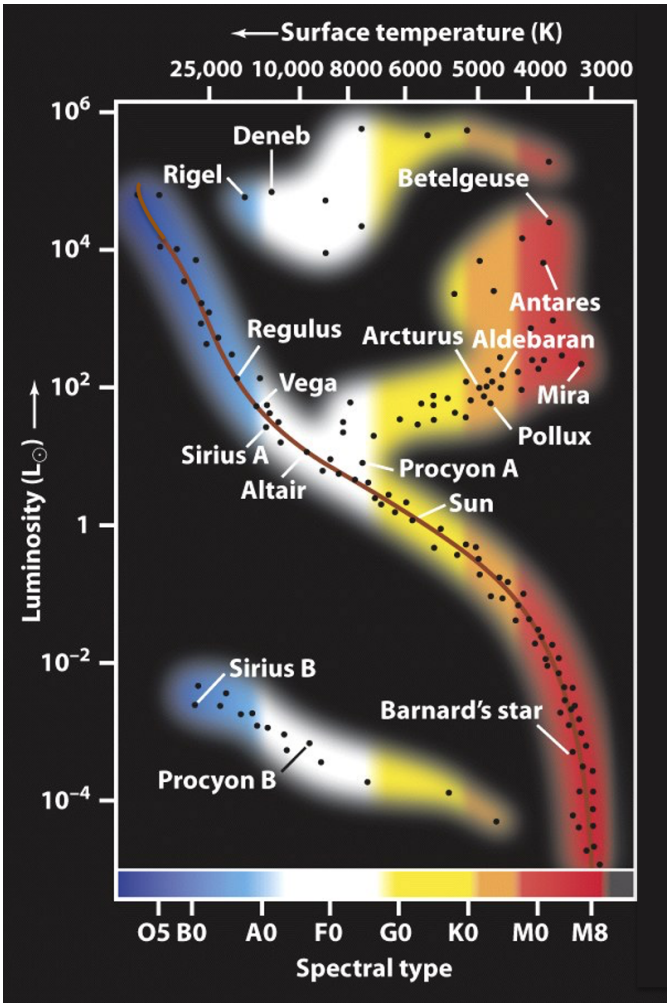
[8]



Section S02 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. (a) Historically, the magnitude system was defined by setting the apparent visual (V-band) magnitude of the brightest star visible unaided in the sky (excluding the Sun) to $m_{V,\mathrm{b}} = 1.0$ and that of the faintest star to $m_{V,\mathrm{f}} = 6.0$. What range of visual fluxes does this correspond to? [2]

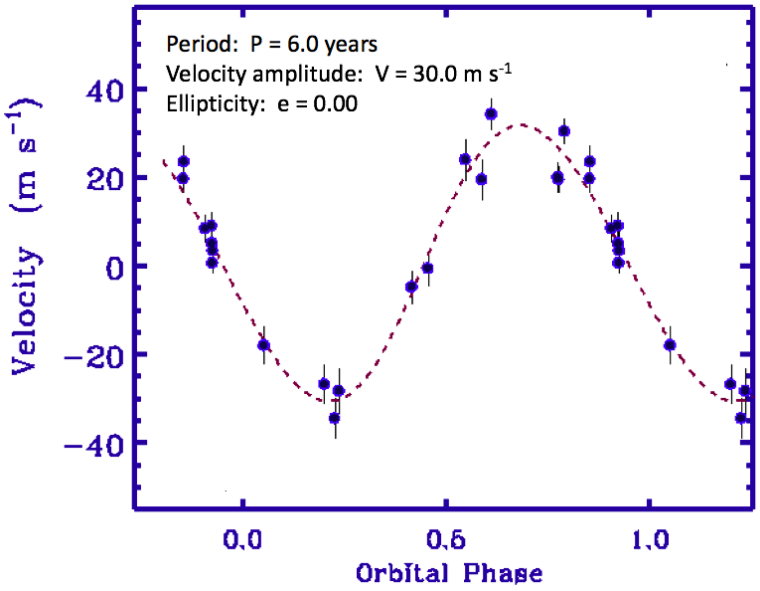
(b) Given the apparent visual magnitude of the Sun $m_{V,\odot} = -26.77$ and the Hertzsprung-Russell diagram below, how far (in parsec) can one see unaided respectively the brightest and faintest main sequence stars? [10]



(c) Using the equations of stellar structure, show the inequality (lower limit) for the central pressure of the Sun $P_{c,\odot} > \frac{GM_\odot^2}{8\pi R_\odot^4}$, where G is the gravitational constant, $M_\odot$ the mass of the Sun, and $R_\odot$ the radius of the Sun. Hence argue that the central temperature of the Sun must be at least several million degrees Kelvin, explaining your reasoning. [9]

(d) Estimate the radius of the star Procyon B. Is it a super-giant, giant, main sequence, or dwarf star? [4]

2. (a) Observations of a Solar-type star reveal both an eclipse in its lightcurve and the radial velocity measurements shown below (best-fit parameters in the figure legend). Determine the mass of the planetary companion and the mean radius of its orbit around the star, explaining your reasoning. [10]



(b) Assume a planet with no atmosphere and no internal energy source orbits a Solar-type star on a circular orbit. Estimate the range of orbital radii of the habitable zone considering only temperature effects. In view of the Solar system, is this radial range reasonable (and why)? [9]

(c) Describe three methods used to detect exoplanets, and discuss for each its main limitation(s)/challenge(s) and bias(es). [6]

[Sun temperature $T_\odot = 5800\,\mathrm{K}$.]

3. (a) The gravitational lens equation is

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm L}\,D_{\rm S}}} \ ,$$

where $\theta_{\rm E}$ is the apparent radius (i.e. the angular radius on the sky) of the Einstein ring, G the gravitational constant, M the total mass contained within the Einstein ring, c the speed of light, $D_{\rm LS}$ the distance between the lens and the source, $D_{\rm L}$ the distance to the lens, and $D_{\rm S}$ the distance to the source.

A galaxy at redshift $z = 0.5$ is lensed by a foreground cluster of galaxies at a redshift $z = 0.2$, yielding an arc of lensed material with a radius of 1 arcminute (60 arcseconds) on the sky. What is the cluster mass (in Solar masses $M_\odot$) within the Einstein ring? [3]

(b) Using the Virial Theorem, derive the expected Fundamental Plane relation of early-type galaxies, clearly stating your assumptions.

The measured Fundamental Plane of early-type galaxies is

$$\log\left(\frac{R_{\rm V,e}}{\rm kpc}\right) = 1.3\log\left(\frac{\sigma_{\rm c}}{\rm km\,s^{-1}}\right) - 0.8\log\left(\frac{I_{\rm V,e}}{\rm mag\,arcsec^{-2}}\right) - 1.36 \ ,$$

where $R_{\rm V,e}$ is the visual (V-band) effective (half-light) radius, $\sigma_{\rm c}$ is the central stellar velocity dispersion, and $I_{\rm V,e}$ is the mean visual (V-band) surface brightness within the effective radius. Why does this measured Fundamental Plane differ from the Virial Theorem expectation? [4]

(c) The same cluster of galaxies as above contains about 100 early-type galaxies within its Einstein ring, where the average galaxy has a Solar-type stellar population, $\sigma_{\rm c} = 250\,{\rm km\ s^{-1}}$, and $I_{\rm V,e} = 21.0\,{\rm mag\ arcsec^{-2}}$. First, estimate the total visual (V-band) luminosity of an average galaxy. Second, estimate the total stellar mass of all these cluster galaxies within the Einstein ring. [10]

(d) Why is the total cluster mass within the Einstein ring so different from the total stellar mass within the same ring?

Adopting a standard cosmology with a ratio of matter density to the critical density of the universe $\Omega_{\rm matter} = 0.30$, similarly for baryonic matter $\Omega_{\rm baryon} = 0.05$, and assuming that most of the baryonic mass in the above cluster of galaxies is in the form of hot (X-ray) gas, calculate the total mass of hot gas in the cluster. Is this consistent with the total cluster and stellar masses previously calculated? Assuming the hot gas is in dynamical and temperature equilibrium, and extends to about twice the cluster Einstein ring, estimate its temperature. Is this estimated temperature reasonable? [8]

[Hubble's constant $H_0 = 75\,{\rm km\ s^{-1}\ Mpc^{-1}}$, absolute visual (V-band) magnitude of the Sun $M_{V,\odot} = 4.8$.]

Section S03 QUANTUM IDEAS

1. (a) Explain briefly what a black body is and discuss its main characteristics. [4]

(b) The energy density of a black body is given by

$$U_\nu = \frac{8\pi h}{c^3}\nu^3 \frac{1}{\mathrm{e}^{h\nu/k_\mathrm{B}T} - 1}.$$

What do the symbols h, ν, c, k_B and T stand for? [2]

(c) Derive the form of the Planck radiation formula in the following limiting cases: $hc/\lambda k_\mathrm{B}T \ll 1$ (large wavelengths) and $hc/\lambda k_\mathrm{B}T \gg 1$ (small wavelengths). Interpret the results. [4]

(d) The temperature of a person's skin is $T_\mathrm{s} = 35\,^\circ\mathrm{C}$. Estimate the peak wavelength of blackbody emission at this temperature. [4]

(e) Estimate the net loss of radiative power by the body in an environment of temperature $T_\mathrm{e} = 20\,^\circ\mathrm{C}$. You may assume that the human skin has a unit emittance in the relevant range and that the surface area of a typical person can be estimated as $A = 2\ \mathrm{m}^2$. [5]

(f) Estimate the net loss of energy during one day. Express the result in calories by use of the conversion relation $1\,\mathrm{cal} = 4.184\,\mathrm{J}$. [3]

(g) Discuss why this result is an overestimate. [3]

2. (a) Explain briefly the Photo-electric effect emphasising why it could not be accounted for within classical electromagnetic theory. [5]

(b) A beam of blue light of wavelength $\lambda = 456\,\mathrm{nm}$ and power $P = 1\,\mathrm{mW}$ is incident on a photosensitive surface of caesium. The electron work function of caesium is $W = 1.95\,\mathrm{eV}$. Determine the maximum velocity of the emitted electrons and the stopping voltage. [4]

(c) If the quantum efficiency of the surface is $\eta = 0.5\%$, determine the magnitude of the photocurrent. The quantum efficiency is defined as the ratio of the number of photoelectrons to that of incident photons. [6]

(d) In an X-ray tube, electrons are accelerated by a potential difference of $V = 25\,\mathrm{kV}$. Determine the minimum wavelength of the X-rays produced when the electrons are stopped at the anode. [5]

(e) Using suitable conservation laws, or otherwise, prove that a free electron cannot absorb a photon. [5]

3. (a) A particle moving on a ring is quantum mechanically similar to a particle in a box. Write down the kinetic energy for a particle of mass m moving in a circle a distance r from the centre in terms of its angular momentum L and the moment of inertia $I = mr^2$. [3]

(b) In terms of the angular variable ϕ, the angular momentum is quantum mechanically given by the following operator:

$$\hat{L} = -i\hbar\frac{\partial}{\partial\phi}$$

and the particle experiences no other external potential.

Set up the time dependent Schrödinger equation for this situation. Show how to obtain the time independent Schrödinger equation and obtain the normalised eigen functions of the particle moving on a ring. You may use the fact that the eigen functions should be single valued, namely that $\Psi(\phi + 2\pi) = \Psi(\phi)$. [12]

(c) Prove that the eigenvalues are given by

$$E_n = \frac{\hbar^2 n^2}{2mr^2}.$$

Explain why this form is reasonable from the perspective of Heisenberg's Uncertainty Principle. [5]

(d) Imagine that this particle jumps from the energy level $n = 3$ to the energy level $n = 2$. What is the wavelength of the photon emitted following this transition? [5]