A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: Mathematical Methods 2

also

Preliminary Examination in Physics and Philosophy

Thursday, 9 September 2021

Opening Time 9.30 am UK time

You have $2\frac{1}{2}$ hours writing time to complete the paper and up to 30 mins technical time to upload your answer file.

Answer **all** *of Section A and* **three** *questions from Section B.*

Write your candidate number, the paper name, paper code and a list of the questions that you have answered on the first page, and submit all of your answers as a single PDF file.

The use of approved calculators is permitted.

A list of physical constants and conversion factors will be available within Inspera during the examination.

The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.

Section A

1. (a) Using series expansions evaluate the limit

$$\lim_{x\to 0} \frac{1+x^2-\exp(x^2)}{x\sin x - x^2} \, .$$

[2]

(b) A function $I(x)$ is defined by the integral

$$I(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \, .$$

By differentiating the integral, compute the first two non-zero terms of the Taylor series of $I(x)$ around $x = 0$. [4]

2. A Red Kite (a type of bird) is flying above Oxford in a vertical circular pattern described by $x = \cos t$, $z = \sin t$, above a heat source. The temperature distribution of the air in this plane is given by

$$T = 20\exp(-x^2) - (z+1)^3$$

Find, in two different ways, the rate of change of temperature that the bird experiences as a function of t. [6]

3. A wave propagating at the speed of light, c, has amplitude $A = A_0\cos(\omega t - kx)$, where A_0 is modulated such that $A_0 = a + b\cos(\omega_m t)$ and where $\omega_m \ll \omega$. The modulation generates two angular frequencies that differ from ω. Derive an expression for A in terms of those frequencies. [4]

4. Find the arc length of the curve given by $x(t) = a(\cos t + t\sin t)$, $y(t) = a(\sin t - t\cos t)$ from $t = 0$ to $t = 2$, where a is a constant. [4]

5. A 3D scalar field is given by $f(x, y, z) = x^2 + 9xy - 2xz^2$. Find the unit vector $\hat{\mathbf{n}}$ along which the derivative of f, evaluated at $(x, y, z) = (2, 4, 8)$ is of greatest magnitude. [4]

6. Evaluate

$$\iint_S x - z \, dS$$

where S is the surface of the solid bounded by $x^2 + y^2 = 4$, $z = x - 3$ and $z = x + 2$. Note that all three surfaces are included in the surface integral. Include a sketch of the surface. [8]

7. A continuous probability density function (PDF) representing the lifespan of a light bulb is given by

$$f_T(t) = \begin{cases} 0.04\exp(-t/25) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where t is in months.

(a) Verify that $f_T(t)$ satisfies the normalisation requirement for a PDF.

(b) Determine the probability, to three significant figures, that the light bulb will have a lifespan of (i) less than 8 months, (ii) more than 20 months, (iii) between 14 and 30 months.

(c) Determine the mean value of the lifespan of the bulbs. [8]

Section B

8. (a) State, carefully describing all the terms, both Stokes' theorem and the Divergence theorem, and discuss their properties and uses. From these theorems, derive a relation which is satisfied by conservative forces. [6]

(b) The vector field $\mathbf{F}$ is given (in polar coordinates) as

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^{\beta}}$$

where $\mathbf{r}$ is the position vector in 3D space, $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$, and $\beta \neq 2$ is a constant. Calculate, by direct integration,

$$\int_V (\nabla \cdot \mathbf{F}) dV$$

where V is the volume between two concentric spherical shells of radii r_1 and r_2 ($r_1 < r_2$). [6]

Verify the divergence theorem for this case by evaluating the appropriate surface integral of $\mathbf{F}$. [4]

(c) Verify Stokes' theorem for $\mathbf{F}$ in part (b), by integrating over the upper hemisphere of the outer spherical shell. [4]

9. (a) A vector field in 2D Cartesian coordinates is given by

$$\mathbf{V}(x, y) = A(x, y)\mathbf{i} + B(x, y)\mathbf{j} \, .$$

Derive the conditions on the functions $A(x, y)$ and $B(x, y)$ such that $\nabla \times \mathbf{V} = 0$. [3]

(b) The line integral

$$I = \int_C \left[(10x^4 - 2xy^3)dx - 3x^2y^2dy\right]$$

is defined along the curve C in the (x,y) plane with start point $(x, y) = (0, 0)$ and end point $(x, y) = (2, 1)$. Show that I is independent of the path taken. Hence, by choosing a simple path connecting the start and end point, evaluate the integral. Indicate your chosen path with a sketch. [7]

(c) Explain the term *exact differential* as it applies to differentials of functions $f(x, y)$. Using a suitable exact differential, evaluate the integral I. [4]

(d) Show that for a closed curve C in the (x, y) plane, the line integral

$$\int_C (xdy - ydx) = 2A$$

where A is the area enclosed by the curve. Hence, by using the change of coordinates $x = a\cos\theta$ and $y = b\sin\theta$, evaluate the area enclosed by the ellipse $(x/a)^2 + (y/b)^2 = 1$. [6]

10. A long uniform string is stretched along the x-axis, has a mass density μ, and is held under tension T. A transverse wave of displacement $y(x,t)$ travels along the string.

(a) The wave equation for transverse waves propagating along the string may be written as

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y(x,t)}{\partial t^2}\,.$$

Show that a Gaussian pulse propagating along the string of form

$$A\exp\left[-(x+vt)^2/x_0^2\right]$$

where A, v and x_0 are constants, is a solution of the wave equation. Sketch the pulse at two different locations corresponding to times t_1 and $t_2 > t_1$. [6]

(b) Show that the wave energy density along the string is given by

$$u = \frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\,.$$

[5]

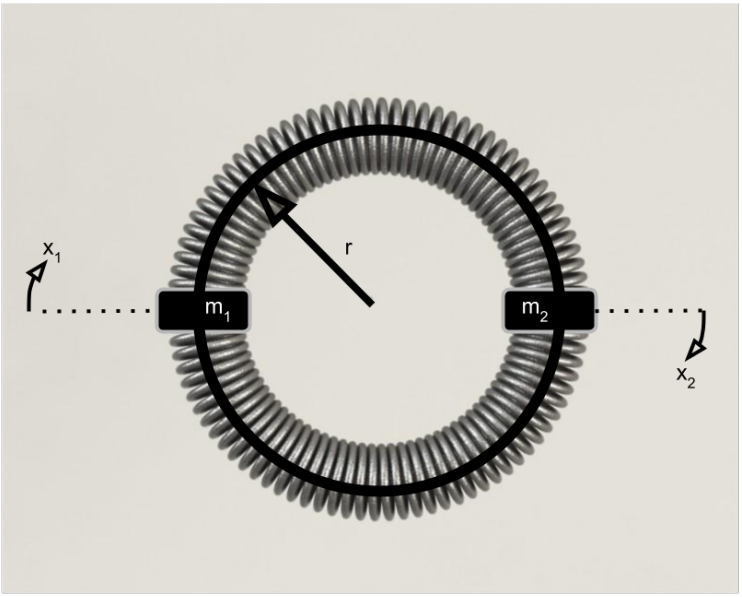
Hence show that for a sinusoidal wave of amplitude A and angular velocity ω travelling along the string, the energy per wavelength is given by

$$E_\lambda = \frac{1}{2}\mu A^2\omega^2\,.$$

[5]

(c) Assume now that the string extends to infinity in the $-x$ direction, but the other end is terminated by a mass m at $x = 0$ which is free to move in the y direction. Calculate the fraction of the energy reflected when a sinusoidal wave propagates in the $+x$ direction. What happens to the rest of the energy? [4]

11. Two masses m_1 and $m_2 = qm_1$ are free to slide on a frictionless horizontal circular wire of radius r (viewed from above in the figure below). The particles are connected by two identical massless springs of spring constant k and natural length πr, which also wind around the wire, as shown in the figure.



(a) Write down the equations of motion for the displacements x_1 and x_2 of the two particles, appropriate for any value of q. [4]

(b) Find the frequencies of the normal modes of the system. [4]

(c) Find the general expressions for the motions associated with each normal mode, including the appropriate amplitudes and time dependence. Describe qualitatively what happens if the mass ratio q goes to infinity. [6]

(d) For this part of the question assume $q = 2$. If, at time $t = 0$, the displacements and velocities of the two particles are given by $x_1(0) = \pi r/10$, $x_2(0) = -\pi r/20$, and $\dot{x}_1(0) = \dot{x}_2(0) = \pi\omega r/20$ (where ω is the largest of the frequencies found in part (b)), find the first time when the velocity of the particle of mass $2m$ is zero. Write your answer in terms of ω. [6]