A10289W1

**FIRST PUBLIC EXAMINATION**

**Trinity Term**

**Preliminary Examination in Physics**

**Paper CP4: Mathematical Methods 2**

**also**

**Preliminary Examination in Physics and Philosophy**

**Thursday, 10 June 2021**

**Opening Time 9.30 am UK time**

**You have $2\frac{1}{2}$ hours writing time to complete the paper and up to 30 mins technical time to upload your answer file.**

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*Answer* **all** *of Section A and* **three** *questions from Section B.*

*Write your candidate number, the paper name, paper code and a list of the questions that you have answered on the first page, and submit all of your answers as a single PDF file.*

*The use of approved calculators is permitted.*

*A list of physical constants and conversion factors will be available within Inspera during the examination.*

*The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.*

## Section A

**1.** Determine the number of stationary points for the functions (i) $6x - 4\cos(3x)$, (ii) $10xe^{3-x^2}$ and (iii) $xe^{x^2}$. [6]

**2.** (i) Let $X$ be a continuous random variable with probability density function (PDF)

$$f_X(x) = \begin{cases} \frac{3}{x^4} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of $X$. [3]

(ii) A two dimensional joint PDF is given by

$$f_{XY}(x, y) = \begin{cases} 6xy & \text{for } 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal PDFs , $f_X(x)$ and $f_Y(y)$. [3]

**3.** A physicist's theoretical model of a fairground ride consists of a rising helix followed by a descending slide with the form of a quarter-circular return to the ground in the (y,z) plane. All units are in metres. The rising helix can be described by the equations

$$x = 10\cos(t - \pi/2)$$

$$y = 10 + 10\sin(t - \pi/2)$$

and

$$z = 5t$$

for $0 \leq t \leq 3\pi$, and the quarter-circular drop takes the ride smoothly from the top of the helix back to the ground. Calculate the total path length of the ride, accurate to three significant figures. [6]

**4.** Find the angle between the surfaces of the sphere $x^2+y^2+z^2 = 2$ and the cylinder $x^2 + y^2 = 1$ at a point where they intersect. [6]

**5.** Evaluate the integral of

$$f(x,y) = \frac{1}{y^{1/3}(x^3+1)}$$

over the region which is bounded by the $x$ axis, the line $x = -y^{1/3}$ and the line $x = 3$. [4]

**6.** Evaluate the line integral of the field $\mathbf{F} = y^2\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ along the curve given by $z = y = e^x$ from $x = 0$ to $x = 1$. [4]

**7.** (i) Derive the wave equation

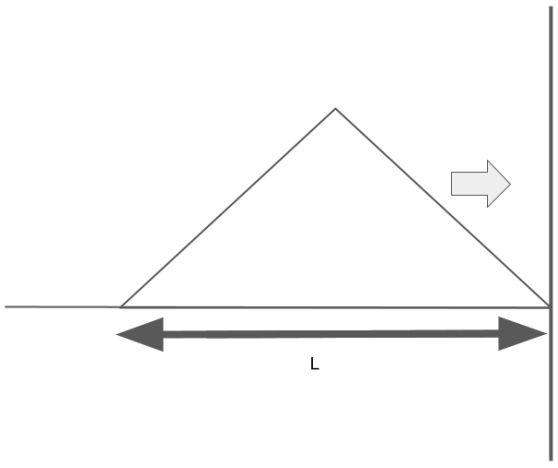
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

explaining the meaning of all terms used.

(ii) Show that the function $y = f(ct - x)$ is a solution of the wave equation. In what direction is this wave moving?

(iii) Show furthermore that the wave profile $y$ remains unchanged with time, by considering an expression for $y$ at a time $t + \Delta t$ where $\Delta t = \Delta x/c$.

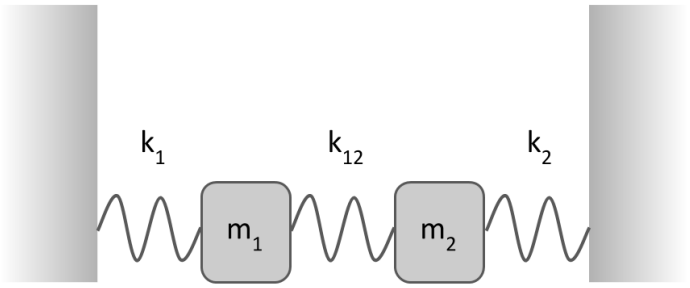
(iv) A triangular shaped pulse of length L is completely reflected at the fixed end of the string on which it travels (see below). Sketch the shape of the pulse after a length (a) L/4, (b) L/2, (c) 3L/4 and (d) L of the pulse has been reflected. [8]



## Section B

**8.** Describe briefly the properties of the *normal modes* of an oscillating system. [2]

A coupled oscillator is sketched in the figure below. It comprises two weights of equal masses $m_1 = m_2$, on a frictionless surface. Mass $m_1$ is connected to a fixed wall by a spring of spring constant $k_1$, and similarly mass $m_2$ is connected to a fixed wall by a spring of spring constant $k_2$. The two weights are coupled by a spring of spring constant $k_{12}$.




Write out the equations of motion for the system in the case that masses $m_1$ and $m_2$ undergo small horizontal displacements $x_1$, $x_2$ from their equilibrium positions. [2]

Calculate the two frequencies of normal modes of oscillation of the system, $\omega_-$ and $\omega_+$, where $\omega_+ > \omega_-$. State any assumptions you make about the form of the general solutions to the equations of motion. Sketch how $x_1$ and $x_2$ will vary as a function of time in the event that both normal modes are excited and are *weakly coupled*. [6]

The oscillator is now *uncoupled* by holding $m_2$ fixed. What is the frequency, $\omega_1$ of oscillation of $m_1$ in this situation if it is displaced from its equilibrium position by a small amount? Similarly, at what frequency, $\omega_2$, does $m_2$ oscillate if now $m_1$ is held fixed and $m_2$ is displaced? [4]

If $k_1 > k_2$, place the four calculated frequencies $(\omega_+, \omega_-, \omega_1, \omega_2)$ in order of ascending frequency. [4]

**9.** (i) State Stokes' theorem, carefully describing all of the terms. For what kind of topology is Stokes' theorem valid? Show how Stokes' theorem applied in a two-dimensional plane becomes Green's theorem. [4]

(ii) Show that

$$\oint_C \mathbf{r} \cdot \mathbf{dr} = 0$$

where $C$ is a closed loop and $\mathbf{r}$ is the position vector. [4]

(iii) Consider how a surface with cylindrical topology can still be used with Stokes' theorem by imagining a vertical (i.e. in the $z$ direction in the cylindrical coordinate system) cut down one of the sides. Sketch such a geometry, with the cut, indicating on each line segment in which direction the line integral should be evaluated.

Use Stokes' theorem to evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{dS}$$

where $S$ is the surface defined by $x^2 + 4y^2 = 1$ and $-1 \leq z \leq 1$ and $\mathbf{F} = xz^2\mathbf{i} - yz^2\mathbf{j}$. [12]

**10.** (i) The divergence of a vector field $\mathbf{F}$ at a point $P$ is defined as the limit of the ratio of the surface integral of $\mathbf{F}$ out of the surface of a closed volume $V$ enclosing $P$ to the volume $V$, as $V$ shrinks to zero. Starting with this definition, and carefully defining all terms, derive the expression for the divergence. Show, stating any assumptions, how this leads to the expression for the divergence theorem for larger volumes. [4]

The surface S is defined by $x^2 + y^2 + z^2 = a^2$.

(ii) For the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$, calculate both the vector surface integral of $\mathbf{F}$ through $S$, and the volume integral of the divergence of $\mathbf{F}$ for the volume enclosed by $S$. Are these consistent with the divergence theorem? [10]

(iii) Using the divergence theorem, determine the flux of the vector field $\mathbf{G} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through $S$. [6]

**11.** (i) Explain what is meant by *dispersion* of waves. Provide three examples of its occurrence in nature, and one (different) example of an application which exploits it. [4]

(ii) Show that the wave equation

$$i\,m\frac{\partial y}{\partial t} = -\frac{\partial^2 y}{\partial x^2}$$

where $m$ is a constant, has complex exponential travelling wave solutions of the form

$$y(x,t) = A\exp[i(kx - \omega t)]\,.$$

Explain the meaning of the parameters $k$ and $\omega$. Show that the *dispersion relation* between $k$ and $\omega$ is $m\omega = k^2$. [4]

(iii) Explain what is meant by *phase* and *group* velocities. For the example given above, find an expression for the phase velocity in terms of $\omega$. [3]

(iv) A travelling wave has two complex exponential components which are equal in magnitude but with frequencies and wavenumbers which differ by a small amount: $\omega = \omega_0 \pm \delta\omega$, $k = k_0 \pm \delta k$. Show that the resultant wave takes the form of a complex exponential travelling wave that is modulated by a slowly varying, real, periodic function. [3]

(v) By considering how $\delta k$ depends upon $\delta\omega$, show that the group velocity of the wave differs from the phase velocity by a factor of two. [3]

(vi) Discuss, aided by a sketch, how the resulting wave form from (iv) would change if a large number of waves were added with frequencies and wave numbers between those of the two original complex exponentials. [3]