A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 8 September 2021

Opening Time 9.30 am UK time

You have $2\frac{1}{2}$ hours writing time to complete the paper and up to 30 mins technical time to upload your answer file.

Answer **all** *of Section A and* **three** *questions from Section B.*

Write your candidate number, the paper name, paper code and a list of the questions that you have answered on the first page, and submit all of your answers as a single PDF file.

The use of approved calculators is permitted.

The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.

Section A

1. Let $z = \frac{1}{\sqrt{2}}(i-1)$. Find the root of $z^{\frac{2}{3}}$ which is furthest from z itself on the Argand plane, showing all your working. [5]

2. Find the real and imaginary parts of

$$i)\ \frac{3+2i}{2-3i}, \qquad ii)\ (1+i\sqrt{3})^3, \qquad iii)\ \left(e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}\right)^4,$$

[5]

showing all your working.

3. (a) Find the angle between the two planes $A : x + 2y - 4z = 5$ and $B : 2x - y + 3z = 8$, showing all your working.

(b) Find another plane C which is parallel to A but which passes through $(4, 2, 1)$. What is the distance between A and C? [6]

4. (a) The columns of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

form a parallelepiped; find its volume.

(b) Now find λ such that the volume corresponding to $\lambda\mathbf{A}$ is the same as the volume corresponding to $\mathbf{A}^5$. [6]

5. Discuss whether the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

is diagonalisable. If so, find the normalised eigenvectors and the matrix $\mathbf{P}$ such that $\mathbf{P^{-1}AP}$ is diagonal. [7]

6. Explain what it means for a matrix to be invertible. Suppose $\mathbf{X}$ is invertible and λ is an eigenvalue of $\mathbf{X}$. Show that $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{X}^{-1}$. [5]

7. What is meant by an *exact differential*? Solve the differential equation

$$6y\frac{dy}{dx} - 2\frac{x^2}{y^3}\frac{dy}{dx} + \frac{2x}{y^2} + x = 0.$$

[6]

Section B

8. (a) Let D be the differential operator $D(f) = \dfrac{df(x)}{dx}$ for a set S of functions $f(x)$. Does D always represent an invertible linear map? Explain your answer using two examples. [5]

(b) Show that the determinant and trace of an $n \times n$ matrix are unchanged by a change of basis. You may use the fact that $\det(AB) = \det(A)\det(B)$. [5]

(c) The set $\{1, t, e^t, te^t\}$ is a basis of a vector space V of functions $f : \mathbb{R} \to \mathbb{R}$. Let F be the differential operator, i.e. $F(f) = \dfrac{df(t)}{dt}$.

(i) Find the matrix of F with respect to this basis.

(ii) The set $\{1-t, 1+t, e^t(1+t), v_4\}$ is also a basis. Find a suitable vector v_4 and determine the matrix representing F with respect to this new basis.

(iii) Does your new matrix conform to the results of part (b)? [10]

9. (i) Suppose three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. Show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$. What is the span of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$? [6]

(ii) Let $\mathbf{x_i} = (x_i, y_i, z_i)$, $i = 0, 1, 2, 3$ be four non-coplanar points in three dimensional space, and let vectors $\mathbf{u_i} = \mathbf{x_i} - \mathbf{x_0}$, $i = 1, 2, 3$ be edges of the tetrahedron formed by those points. Consider the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ with magnitudes in turn equal to twice the area of the four faces of the tetrahedron, and directions outwards and normal to those faces. Express these vectors in terms of the $\mathbf{u_i}$, and hence show that $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$. [8]

(iii) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in three dimensional real space.

(a) Show that in the scalar triple product the 'dot' and the 'cross' can change places, by demonstrating that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

(b) Find, in terms of $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$, the value of the scalar triple product $\langle \mathbf{U}, \mathbf{V}, \mathbf{W} \rangle$, where

$$\mathbf{U} = \frac{\mathbf{v} \times \mathbf{w}}{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle}, \qquad \mathbf{V} = \frac{\mathbf{w} \times \mathbf{u}}{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle} \quad \text{and} \quad \mathbf{W} = \frac{\mathbf{u} \times \mathbf{v}}{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle},$$

showing all your working. [6]

10. An operator $\mathbf{L}$ acts on a function $f(t)$ such that

$$\mathbf{L}f = a_2\frac{d^2f}{dt^2} + a_1\frac{df}{dt} + a_0 f = g,$$

where the a_i are real constants and $g = g(t)$.

(a) Under what conditions do solutions f to $\mathbf{L}f = 0$ have oscillatory components? [4]

(b) Solve for the particular integral when $g(t) = 8-4t+3t^2+2t^3$ with $(a_0, a_1, a_2) = (2, 3, 1)$. [5]

(c) Solve for the particular integral when $g(t) = 2e^t$, with $(a_0, a_1, a_2) = (1, -2, 1)$. [5]

(d) With $g(t) = A\cos 2t$ and $(a_0, a_1, a_2) = (1, 1, 1)$, what is the limiting behaviour of $f(t)$ as $t \to \infty$? [6]

11. Find the general solutions to the following differential equations, showing all your working:

(a)

$$x^2\frac{dy}{dx} = y^2 - xy,$$

[5]

(b)

$$2(1+x)\frac{dy}{dx} - (1+2x)y = x^2(1+x)^{1/2},$$

[10]

(c)

$$\frac{dy}{dx} + \frac{y}{x} = xy^2.$$

[5]