A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday, 9 June 2021

Opening Time 9.30 am UK time

You have $2\frac{1}{2}$ hours writing time to complete the paper and up to 30 mins technical time to upload your answer file.

Answer **all** *of Section A and* **three** *questions from Section B.*

Write your candidate number, the paper name, paper code and a list of the questions that you have answered on the first page, and submit all of your answers as a single PDF file.

The use of approved calculators is permitted.

The numbers in the margin indicate the weight that the examiners expect to assign to each part of the question.

Section A

1. Sketch the locus of $z = e^{i\theta}\sin\theta$ in the complex plane as θ varies from 0 to 2π, and describe the shape of the curve near the origin. [4]

2. Solve the equation $2\sin ix - 3i\cos ix = 3i$, demonstrating your method. [5]

3. (a) An upper triangular matrix satisfies $a_{ij} = 0 \;\forall\; i > j$. If $\mathbf{A}$ and $\mathbf{B}$ are both upper triangular $n \times n$ matrices, prove that $\mathbf{AB}$ is upper triangular and find its diagonal elements.

(b) Find an upper triangular matrix $\mathbf{A}$ such that

$$\mathbf{A}^3 = \begin{bmatrix} 27 & -74 \\ 0 & 64 \end{bmatrix}.$$

[6]

4. Find all possible solutions of the set of equations

$$\begin{aligned} x + y - z &= 1 \\ 2x + 2y + \lambda z &= 2 \\ x - \lambda y &= 1. \end{aligned}$$

Hence determine whether there is a value of the parameter λ such that there is a solution in which both x and y are negative. [7]

5. (a) The matrix

$$\mathbf{P} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

relates coordinates $\boldsymbol{\alpha'}$ in basis $\{\mathbf{e_1}', \mathbf{e_2}'\}$ to coordinates $\boldsymbol{\alpha}$ in the orthonormal basis $\{\mathbf{e_1}, \mathbf{e_2}\}$ according to $\boldsymbol{\alpha'} = \mathbf{P}\boldsymbol{\alpha}$. Explain this transformation by including a sketch illustrating both sets of axes and coordinates for an example point. [4]

(b) Find the matrix $\mathbf{A}$ which performs a reflection about the line $y = \frac{x}{2}$ in the basis $\{\mathbf{e_1}, \mathbf{e_2}\}$. What is $\det \mathbf{A}$ and how do you interpret this? [4]

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Show that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues. [4]

7. Solve the equation

$$\frac{dy}{dx} = \frac{4x + 2y - 10}{2x - y - 3}$$

by first making the change of variables: $x = X + a$, $y = Y + b$. Explain why this method works, by referring to the lines $4x + 2y - 10 = 0$ and $2x - y - 3 = 0$. [6]

Section B

8. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

(i) Describe the vector space spanned by $\mathbf{a}$ and $\mathbf{b}$ and give its dimension depending on the properties of the two vectors.

(ii) Prove that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. [7]

(b) Show that the set of vectors in $\mathbb{R}^3$ orthogonal to a vector $\mathbf{v}$ form a vector subspace. Does this mean that every plane forms a vector subspace of $\mathbb{R}^3$? [4]

(c) Let $\mathcal{U}$ and $\mathcal{V}$ be planes which are vector subspaces of $\mathbb{R}^3$. State an equation relating $\dim(\mathcal{U} + \mathcal{V})$ and $\dim(\mathcal{U} \cap \mathcal{V})$ and determine their possible values. [4]

(d) Now suppose additionally that $\mathcal{U}$ passes through the points $(1, 1, 3)$ and $(1, 3, 1)$, and that $\mathcal{V}$ passes through $(1, 1, -2)$ and $(2, 1, 2)$. Find the shortest distance from the point $(1, 1, 1)$ to the intersection of the planes. [5]

9. (a) Let $\mathbf{M}$ be a Hermitian matrix. Prove that the matrix $e^{i\mathbf{M}}$ is unitary by writing it as a power series. [6]

(b) Let $\mathbf{A} = (a_1, a_2, \dots)$ be an infinite series of real numbers such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. The set of all such series forms a vector space V with addition and scalar multiplication defined by

$$\alpha\mathbf{A} + \mathbf{B} = (\alpha a_1 + b_1, \alpha a_2 + b_2, \dots).$$

Show that $\langle \mathbf{A}, \mathbf{B} \rangle = a_1 b_1 + a_2 b_2 + \dots$ defines a scalar product in V. Further, show that the series defined by this scalar product converges for any pair of vectors in V. [5]

(c) G_i is a series defined by $G_{i+2} = (G_i + G_{i+1})/2$, with $G_0 = 0$ and $G_1 = 1$. By expressing this definition in the form

$$\begin{pmatrix} G_{i+2} \\ G_{i+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} G_{i+1} \\ G_i \end{pmatrix},$$

where $\mathbf{A}$ is a 2×2 matrix, show that $G_i \to 2/3$ as $i \to \infty$. [9]

10. (a) Solve the differential equation

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0.$$

[Hint: one approach is to introduce the parameter $p = dy/dx$ and then show that $dp/dx = p/x$.]
How many integral curves pass through each point of the (x, y) plane (except $x = 0$)? Why is the solution at each point not unique? [11]

(b) The differential equation

$$\left(\frac{dy}{dx}\right)^2 + f(x,y)\frac{dy}{dx} - 1 = 0$$

represents a set of curves such that two curves pass through any given point. Show that these curves intersect at right angles at the point. For $f = -2y/x$, verify this property for the point (3, 4). [9]

11. (a) Given the differential equation

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = D,$$

what forms could the functions A, B, C and D take for an integrating factor to be sufficient as a method of solution? How is the integrating factor calculated? [6]

(b) The coordinates (x, y, z) of a particle's location satisfy $x + y + z = t^2$ and also the differential equations

$$\frac{dx}{dt} + 2\frac{dz}{dt} = 3x - y \qquad\qquad 2\frac{dy}{dt} - \frac{dx}{dt} = y - x.$$

Solve for the location as a function of time t, given the initial conditions $x = 2$ and $y = 8/3$ when $t = 0$. Can this motion be maintained by a constant force? [14]