A14119W1

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SHORT OPTIONS

TRINITY TERM 2019

Tuesday, 18 June
9.30 am to 11.00 am

Answer **two** *questions from the option for which you have entered.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

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Some sections may start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. (a) Let $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$, where u and v are real functions. If f is analytic, show that u and v satisfy the Cauchy–Riemann conditions and the curves of constant u are orthogonal to curves of constant v. [6]

(b) Prove that

$$u = \mathrm{e}^{-2xy}\sin(x^2 - y^2)$$

is a harmonic function. More generally, show that if f is analytic then u and v are harmonic functions. Find the corresponding v that makes f analytic. Find an expression for f in terms of $z = x + \mathrm{i}y$. [9]

(c) For $z = r\mathrm{e}^{\mathrm{i}\theta}$ show that the Cauchy–Riemann conditions can be rewritten as

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.$$ [4]

(d) Let the function $g(x,y)$ be given by

$$g(x,y) = \mathrm{e}^{-\theta}\cos(\ln r) + \mathrm{i}\,\mathrm{e}^{-\theta}\sin(\ln r).$$

Show that g is analytic and find $\mathrm{d}g/\mathrm{d}z$. [6]

2. Consider the functions

$$f(z) = (z\mathrm{e}^{\frac{1}{z}})^n \quad \text{and} \quad g(z) = f(z) - f(-z),$$

where n is a positive integer.

(a) Find and classify the singularities of $f(z)$. [6]

(b) Find the Laurent series for $f(z)$ around $z = 0$. [5]

(c) Evaluate the integral of the function $f(z)$ around the circle of centre $z = 0$ and arbitrary radius. [7]

(d) Evaluate the integral of the function $g(z)$ around the circle of centre $z = 1$ and radius 2. [7]

3. (a) State Cauchy's Theorem. [3]

(b) Use contour integration to evaluate

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,\mathrm{d}\theta.$$ [6]

(c) Find the singularities of

$$f(z) = \frac{z^4}{\sqrt{z^2 - 1}}$$

and find two possible sets of branch cuts. [5]

(d) Evaluate the integral

$$\int_{C_\infty} f(z)\,\mathrm{d}z.$$

where C_∞ is the circle of infinite radius. [6]

(e) Choose a contour that encloses a branch cut to evaluate

$$\int_{-1}^{1} \frac{x^4}{\sqrt{1 - x^2}}\,\mathrm{d}x.$$ [5]

Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. (a) State Kepler's third law of planetary motion and write it in equation form in either S.I. units or other convenient units which you should define. [3]

(b) An eclipsing spectroscopic binary star has a period of 20 days. Radial velocity measurements show the two stars moving with velocity amplitudes of $60\,\mathrm{km\,s^{-1}}$ and $120\,\mathrm{km\,s^{-1}}$. Draw a diagram of this binary system labelling the masses, velocities and separation of the stars and including the direction from which it is viewed. What is the separation of the stars in Astronomical Units? Calculate the masses of the stars in units of the solar mass. [9]

(c) The Main Sequence lifetime of the Sun is 10^{10} years. Estimate the Main Sequence lifetime of both stars in the eclipsing binary. The final state of the evolution of the more massive star is a black hole. Estimate the fraction of the rest mass that will be radiated away when a mass m falls towards a black hole and ends up in a stable orbit at a radius of three Schwarzschild radii (you may use the Newtonian expression for the decrease in potential energy). Describe what happens to the system when the lower mass star ceases core hydrogen burning. [13]

2. (a) Derive the equation of hydrostatic equilibrium relating the pressure, $P(r)$, to the interior mass, $M(r)$, and the density, $\rho(r)$, at radius r in a star. Use this to make an estimate of the central pressure in the Sun. By assuming that the plasma in the interior of the Sun obeys the ideal gas law, $P = \rho k_{\mathrm{B}} T/\mu M_{\mathrm{p}}$, where μ is the mean molecular weight in proton masses, M_{p}, make an estimate of the central temperature of the Sun. [7]

(b) By considering the free-fall time from the edge to the centre of the Sun, estimate the dynamical timescale on which the Sun must restore hydrostatic equilibrium if gravity and pressure are out of balance. Show that the minimum total potential energy of the Sun, Ω, is given by

$$|\Omega| > \frac{GM_{\odot}^2}{2R_{\odot}}.$$

Use the virial theorem to estimate the lifetime of the Sun if gravitational contraction is the only source of energy. [9]

(c) Draw the Hertzsprung–Russell diagram for a globular cluster labelling the axes and the main branches populated by stars. The stars at the Main Sequence turn off point of a globular cluster have a luminosity of $1.6 \times 10^{26}\,\mathrm{W}$ and a mass of $0.8\,M_{\odot}$. The core of these stars is burning hydrogen to helium with 0.7% efficiency. Estimate the age (in years) of the globular cluster stating the assumptions you make. [9]

3. (a) Describe the components of the structure of the Milky Way giving the approximate mass and size and the main constituents of each component. [10]

(b) The Sun resides in the disk of the Milky Way 8 kpc from the centre and moves on a circular orbit with speed $220\,\mathrm{km\,s^{-1}}$. Calculate the mass of the Milky Way out to 8 kpc in solar masses. We estimate the luminosity of the Milky Way within 8 kpc to be $2 \times 10^{10} L_{\odot}$. By assuming that within this radius the luminous material accounts for all of the mass, make an estimate of the mass of an average star in the Milky Way. [4]

(c) A cluster of galaxies contains 1000 galaxies within a sphere of radius 1 Mpc. The root-mean-square radial velocity of the galaxies is measured to be $900\,\mathrm{km\,s^{-1}}$ and the rotation of the cluster is negligible. Make an estimate of the total mass of the cluster in solar masses. The cluster contains X-ray emitting gas that has a temperature, $T = 2{\times}10^{8}\,\mathrm{K}$, that also extends to 1 Mpc. Make a second estimate of the mass of the cluster. By assuming each galaxy has a luminosity of $5 \times 10^{9} L_{\odot}$, and using the mass-to-light ratio for the Milky Way which you derived above, make an estimate of the total stellar mass of the galaxies in the cluster. [8]

(d) Give three reasons that could explain why the sum of the stellar masses is much less than the total cluster mass derived from dynamics or X-rays. [3]

Section S3 QUANTUM IDEAS

1. (a) Explain briefly what a black body is and discuss its main characteristics. [4]

(b) The energy density emitted from a black body is given by

$$U_\nu = \frac{8\pi h}{c^3}\nu^3 \frac{1}{\mathrm{e}^{h\nu/k_\mathrm{B}T} - 1}.$$

What do the symbols h, ν, c, k_B and T stand for? Plot the energy density as a function of frequency and shade the area representing the total energy output of the black body. Without performing any detailed integration, comment on how the total emitted energy scales with temperature. [6]

(c) Derive the energy density in the high-temperature limit, where the photon energy is much smaller than $k_\mathrm{B}T$. Now derive the result at the other extreme, when the temperature is low compared to $h\nu$. Comment on the difference between the two in the light of Planck's quantum hypothesis. [7]

(d) Wien's displacement law states that the wavelength at which the energy density is highest is given by

$$\lambda_\mathrm{max} = \frac{0.002897\,\mathrm{K\,m}}{T}\,.$$

Explain briefly, and without any extensive derivations, how you would obtain this result. [3]

(e) Given that the colour of the sun is yellow, corresponding to a characteristic wavelength of 570 nm, use Wien's displacement law to estimate the surface temperature of the Sun. The core of the Sun is at the temperature of more than 20 million degrees. Comment on what could account for this difference between the surface and the core. [5]

2. (a) Explain briefly the Compton scattering effect. Draw a diagram of the Compton scattering process clearly labelling every arrow as well as the angle of scattering and deflection. [4]

(b) Write down the momentum and energy conservation equations for Compton scattering, in the case where the electron motion is non-relativistic. [5]

(c) Assume that the photon back scatters fully after the collision. Solve the conservation equations to obtain the formula for the photon's wavelength change, $\Delta\lambda$, in the process. You may assume that $\Delta\lambda$ is much smaller than the wavelengths of the photon before and after the scattering. [8]

(d) What is the Compton wavelength and what is its physical significance? [3]

(e) What is the speed of the electron after the collision, if the wavelength of the incoming photon is 10 nm? Explain therefore why we are allowed to treat the electron non-relativistically as far as the calculation is concerned. [5]

3. (a) Solve the time independent Schrödinger equation for a particle of mass m existing in a one dimensional infinitely deep square well, $0 \leq x \leq L$, to obtain all the allowed eigenstates and eigenenergies. [8]

A wave function for a particle mass m in one dimension is given by

$$\Psi(x) = Nx$$

in the region $0 \leq x \leq L$.

(b) Determine the normalization constant N. [4]

(c) What is the probability that the particle is measured in the region between 0 and $L/3$? [4]

(d) What, on the other hand, is the probability to find the particle in the ground state of the square well? [6]

(e) Suggest an intuitive explanation as to why the latter probability is greater than the former. [3]