A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: Mathematical Methods 2

also

Preliminary Examination in Physics and Philosophy

Thursday 5 September 2019, 9.30 am – 12.00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. A travelling wave on a horizontal string obeys the following equation:

$$y = 0.27 \sin[12x - 500t]\,,$$

where the vertical displacement y, in mm, is given as a function of distance x in metres and time t in seconds. Find the equations for the transverse velocity and acceleration of a small element of string at $x = 20$ cm. What is the displacement of this element when $t = 4$ seconds? For all parts of this question clearly state the units of your answers. [4]

2. Find the tangent plane to the surface $x^2y = 4z\exp[x + y] - 11$ at (2,-2,9). [4]

3. The exponential probability density function (PDF) is given by:

$$f_X(x) = \begin{cases} \lambda \exp[-\lambda x] & \text{for } x \geq 0 \\ 0 & \text{for } x < 0\,. \end{cases}$$

Derive the cumulative distribution function (CDF), the expected value, and variance of the exponential distribution. [6]

4. Suppose the temperature in (x, y, z) space is given by $T = xyz$. Calculate the mean temperature (i.e. temperature per unit volume) in cubes which have corners at
(i) (0,0,0) and (2,2,2) ;
(ii) (0,0,0) and (3,3,3) .
For a cube with corners at (0,0,0) and (s,s,s), derive an expression for how the mean temperature depends upon s. [6]

5. Describe what is meant by vector field which is *conservative*. Describe a test for such a field, and determine whether the vector field

$$\mathbf{F} = (2x\exp[xy] + x^2y\exp[xy])\mathbf{i} + (x^3\exp[xy] + 2y)\mathbf{j}$$

is conservative.
Describe what is meant by *exact* and *inexact* differentials. Give an example of an exact differential, and show that the differential $(x\,\mathrm{d}y + 42y\,\mathrm{d}x)$ is inexact.
What is the relation between conservative vector fields and exact or inexact differentials? [4]

6. State Stokes' theorem, clearly describing all the terms.

The surface S is that part of the plane $x + y + z = 1$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$), oriented with the upward-pointing normal. It has a boundary C, oriented counter-clockwise when viewed from above. Taking the vector field

$$\mathbf{F} = (x^2 - y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + (z^2 - x^2)\mathbf{k}\,,$$

verify Stokes' theorem by computing both

$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,,$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{dS}\,.$$

[10]

7. Consider two identical pendula (equal masses m and string lengths l) initially hanging vertically from a horizontal bar, all of which is in a plane S. The masses are connected by a horizontal massless spring of spring constant k. Show that the two normal modes of oscillation of the pendula in the plane S have angular frequencies

$$\omega_1 = \sqrt{\frac{g}{l}}\,,$$

and

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}\,,$$

where g is the acceleration due to gravity. [6]

Section B

8. (a) State the divergence theorem, defining all terms clearly. Show that for any closed surface S in a vector field $\mathbf{F}$,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = 0\,.$$

[4]

(b) The electric field from a positive point charge placed at the origin is given by $\mathbf{E} = A\hat{\mathbf{r}}r^{-2}$ where $\mathbf{r}$ is the position vector and $A = Q(4\pi\epsilon_0)^{-1}$. Calculate

(i) the divergence of $\mathbf{E}$,

(ii) the surface integral of $\mathbf{E}$ over a sphere of radius R centred on the origin.

Comment on your results in terms of the divergence theorem.

Describe the properties of the Dirac delta function. How may its use simplify the representation of the divergence of the electric field? [8]

(c) The surface S is that part of the sphere of radius 4, centred at the origin, which lies in the region $z \leq 0$, $y \leq 0$. Sketch S, and evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{dS}\,,$$

where $\mathbf{F} = yx^2\mathbf{i} + (xy^2 - 3z^4)\mathbf{j} + (x^3 + y^2)\mathbf{k}$. [8]

9. (a) Explain what is meant by a *stationary point*. Find the stationary points of the functions:

(i) $f(x) = 15 - (3 - x)[x^2 - 8x + 7]^{1/3}$;

(ii) $g(y) = -y + 4\cos y$. [8]

(b) Find the MacLaurin series for $\ln(1 + x)$ and hence evaluate

$$\lim_{x\to\infty} \left(1 - \frac{a^2}{x^2}\right)^{x^2} .$$

[6]

(c) Find an exact expression for the Taylor series for

$$f(x) = \frac{7}{x^4}$$

around the point $x = -3$. [6]

10. (a) Starting with the chain rule, derive a general expression for the Jacobian used in transforming between two general 3D curvilinear coordinate systems (a, b, c) and (u, v, w) (you may represent this as the determinant of matrix).

Using the above method, explicitly calculate the Jacobian associated with the transformation from 3D cartesian coordinates (x, y, z) to

(i) cylindrical coodinates (r, ϕ, z), and

(ii) spherical polar coordinates (r, θ, ϕ). [5]

(b) A solid cylinder of length 4 and radius 1 has a (2D) radial density profile

$$\rho = \rho_0 r^{-1} \,,$$

where r is the radial distance from the axis of the cylinder.

A sphere of radius 1 has a (3D) density profile

$$\rho = \rho_0 r^n \,.$$

In both cases ρ_0 is a constant. For what value of n is the mass of the sphere the same as that of the cylinder? [5]

(c) Describe what is meant by a *line integral*, and the similarities and differences between vector and scalar line integrals. [2]

A closed path C is described by the upper half ($y \geq 0$) of the circle of radius 1 (in the xy plane) centred at the origin, and the portion of $y = x^2 - 1$ between $-1 \leq x \leq +1$, all of which is traversed in an anti-clockwise direction. Sketch C, and evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr} \,,$$

where $\mathbf{F} = 3y\mathbf{i} + (x^2 - y)\mathbf{j}$. [8]

11. (a) Find the resultant disturbance in a long wire under tension when two transverse waves, of equal amplitudes and frequencies and travelling in opposite directions, pass through each other. Discuss your result.

Show that two superimposed waves of the same (non-zero) frequency and amplitude travelling in the same direction cannot give rise to a standing wave. [6]

(b) What is the relationship between the wave number k and angular frequency ω of an oscillation on a string and the tension T and linear density ρ of the string?

Two semi-infinite horizontal strings of different linear densities are joined at $x = 0$ (i.e. one string extends to the left, from $-\infty < x \leq 0$ and the other string extends to the right, from $0 \leq x < +\infty$). An oscillation of amplitude A propagates from the left; at the boundary there is a reflected wave of amplitude B and a transmitted wave of amplitude C.

Show that the coefficients of reflection and tranmission are given by

$$r \equiv \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$t \equiv \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

where k_1 and k_2 are the wave numbers of the oscillations to the left and right of the join, respectively. [6]

(c) Consider now the case that there is a mass M of infinitesimal physical size at the boundary, joined to each string. Show that the coefficients of reflection and transmission are now given by:

$$r \equiv \frac{B}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M}$$

$$t \equiv \frac{C}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M}$$

in terms of the string tension T, the mass M, the angular frequency of oscillation ω and the wavenumbers k_1, k_2 to the left and right of the mass respectively. [8]