A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 13 June 2019, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find an exact expression for the Taylor series for

$$f(x) = e^{-6x}$$

about $x = -4$. [4]

2. Derive an expression for the solid angle of a cone with half-opening angle θ, as well as its small-angle approximation.
Both the Sun and the Moon are seen from the Earth with an average angular diameter of 0.533 degrees. Calculate the solid angle subtended by the Sun or the Moon, and calculate what fraction this is of the entire celestial sphere. [4]

3. Explain why the function

$$f(x) = x^2 \ln(3x) - 1$$

only has one stationary point, and determine its value. [4]

4. State the relationship between the probability density function (PDF) and the cumulative distribution function (CDF) of a continuous random variable. Let X be a continuous random variable with PDF given by

$$f_X(x) = \frac{1}{2} e^{-|x|} ,$$

for $-\infty \leq x \leq +\infty$. If $Y = X^2$, find the CDF for Y. [6]

5. Using the chain rule, or otherwise, differentiate

$$f(x) = \left((8x)^{1/2} + \sin^2(2x) \right)^{-1} .$$

[4]

6. Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{dS} ,$$

where

$$\mathbf{F} = \sin(\pi x)\mathbf{i} + zy^3\mathbf{j} + (z^2 + 4x)\mathbf{k} ,$$

and S is the surface of the box with $-1 \leq x \leq 2$, $0 \leq y \leq 1$ and $1 \leq z \leq 4$. All six sides of the box are included in S. [8]

7. Show that the average power transmitted by a sinusoidal wave in a string is

$$P = \frac{1}{2} \omega^2 A^2 \rho v ,$$

where A is the amplitude of the wave, ω is the angular frequency, ρ is the density (mass per unit length) and v is the speed of the wave. [6]

A 120 Hz wave on a string has an amplitude of 0.16 mm. How much energy exists in a 80 g length of the string? You may assume that one wavelength of the string has a mass far smaller than 80 g. [4]

Section B

8. (a) A solid body rotates around a given axis with angular velocity ω. The vector field describing the velocity at any point in the body is given by $\mathbf{v}$. Derive the relationship between $\nabla \times \mathbf{v}$ and the angular velocity. [4]

(b) State Stokes' theorem, defining all terms clearly. Explain the difference between surfaces which are simply-connected and those which are not. Provide an example of each type of surface, and state for which type of surface Stokes' theorem is valid. [4]

(c) Consider the vector field given by

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{(x^2 + y^2)^n}\,.$$

Note that $\nabla \times \mathbf{F}$ indicates in which direction an infinitesimally small sphere would rotate in a flow described by $\mathbf{F}$.

(i) Calculate $\nabla \times \mathbf{F}$ as a function of n.

(ii) For which values of n is the direction of rotation of the sphere in the same sense, opposite sense, or stationary, compared to the large-scale circulation of the flow?

(iii) For the case n=1, rewrite the vector field $\mathbf{F}$ in terms of the azimuthal unit vector $\hat{\theta}$ and evaluate the line integral of this field in a circular loop of radius r around the origin. How does the value of this line integral vary with r?

Discuss how consistent these results are with Stokes' theorem. [12]

9. (a) (i) A string of uniform linear density ρ is stretched to a tension T. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that the following wave equation is satisfied

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}\,,$$

and find an expression for the wave speed c. [6]

(ii) A flexible steel cable of total length L and mass per unit length μ hangs vertically from a support at one end. Find an expression for the speed of a transverse wave along the cable in terms of L, the acceleration due to gravity g and the distance x along the cable from the support point. Determine how long it will take such a wave to travel down the full length of the cable. [8]

(b) State what is meant by a dispersive medium and what is the dispersion relation. Defining all terms, write expressions for the *phase velocity* and the *group velocity* and explain carefully what travels at each velocity.

The dispersion relation for guided electromagnetic waves in a waveguide (with a non-zero longitudinal component of either E or B) is given by

$$\omega^2 = \omega_0^2 + c^2 k^2\,,$$

where c is the speed of light. Evaluate the product of the phase and group velocities, and show that it implies that one of the velocities must exceed the speed of light. Discuss briefly why this does *not* allow signalling at faster than the speed of light. [6]

10. (a) Sketch the cylindrical and spherical polar coordinate systems and give expressions for the cartesian components x, y, z in each system. [4]

(b) Consider a thin hemispherical shell described by the upper half of the sphere $x^2 + y^2 + z^2 = 1$. The surface density (mass per unit area) is given by $\sigma(x, y, z) = z$. What is the total mass of the shell? [6]

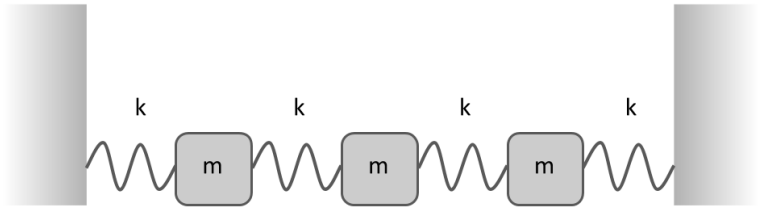
(c) Evaluate

$$\iint_S y + z \, dS \,,$$

where S is the surface whose side is the cylinder $x^2 + y^2 = 3$, whose bottom is the disc $x^2 + y^2 \leq 3$ in the xy-plane and whose top is the plane $z = 4 - y$. [10]

11. Describe what is meant by a normal mode. Give two examples of common systems exhibiting normal modes. [4]

Three identical masses m sit on a frictionless surface, and are connected in a line by identical massless springs of spring constant k and also to fixed walls at either end, as in the figure below.



(a) Determine the equations of motion for the system in the event that the masses are displaced by small amounts from their equilibrium positions. [6]

(b) Calculate the frequencies and eigenvectors of the normal modes of oscillation of this configuration. [6]

(c) Sketch the initial motions of the masses for each of the normal mode frequencies calculated in part (b). Which of the normal mode frequencies determined in (b) would not change if the central mass were changed for one much more massive than the other two? [4]