A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 4 September 2019, 9.30 am – 12.00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is permitted.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) What complex number multiplied by $(1-i)^3$ gives i ?
(b) Write $(-e)^{i\pi/2}$ in the form $a+b\,i$. [4]

2. Find the locus of points on the Argand plane satisfying

$$\arg(z-i^3)=\frac{\pi}{4}.$$

[4]

3. Find the inverse of the matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{pmatrix}.$$

[5]

4. An upper triangular matrix $\mathbf{M}$ is defined by having elements $m_{jk}=0$ for all $k<j$. Show that the determinant of $\mathbf{M}$ is the product of its diagonal elements. [5]

5. Write down the vector equation of the line

$$\frac{(x-1)}{2}=\frac{(y-1)}{3}=\frac{(z-1)}{4},$$

and find the shortest distance to the origin. [4]

6. The linear equations

$$\begin{aligned} x+2y+3z &= 2 \\ \alpha y+z &= \beta \\ 2x+2y &= 1 \end{aligned}$$

have three possible solutions depending on α and β. Find these solutions and the constraints placed upon the constants α and β. [6]

7. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+3\tan\left(\frac{y}{x}\right).$$

[5]

8. Find the solutions $x(t)$ and $y(t)$ of the following simultaneous differential equations, subject to the initial conditions $x=1$ and $y=1$ at $t=0$:

$$\frac{\mathrm{d}x}{\mathrm{d}t}+y=1$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}-x=1\,.$$

[7]

Section B

9. What is a condition that ensures that three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar? [4]

Two particles have position vectors at time t given by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ where

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{a}_1 + \mathbf{b}_1\,(t - t_1) \\ \mathbf{r}_2 &= \mathbf{a}_2 + \mathbf{b}_2\,(t - t_2)\,. \end{aligned}$$

Find the shortest distance between the particles. [4]

Show that, at the minimum separation, $(t_1 - t_2)$ is given by

$$(t_1 - t_2) = \frac{(\mathbf{a}_2 - \mathbf{a}_1).\,[(\mathbf{b}_1 \times \mathbf{b}_2) \times (\mathbf{b}_1 - \mathbf{b}_2)]}{(\mathbf{b}_1 \cdot \mathbf{b}_2)^2 - (\mathbf{b}_1 \cdot \mathbf{b}_1)(\mathbf{b}_2 \cdot \mathbf{b}_2)}\,. \qquad [12]$$

10. Find the general solution to each of the following equations:

(a) $\dfrac{dy}{dx} + 2y = e^{-2x} + x^2 + 2x\,,$ [6]

(b) $y\dfrac{dy}{dx} = \dfrac{x}{x+1}\,,$ [6]

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 3x^2e^{-x}\,.$ [8]

11. (a) Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian operators, while $\mathbf{C}$ is a non-Hermitian operator. Explain which of the following operators are also Hermitian.

(i) $i(\mathbf{AB} - \mathbf{BA})$,

(ii) $(\mathbf{AB} - \mathbf{BA})$,

(iii) $\frac{1}{2}(\mathbf{AB} + \mathbf{BA})$,

(iv) $\mathbf{C}^\dagger\mathbf{C}$. [6]

(b) The Pauli matrices $\boldsymbol{\sigma}_i$ are defined as

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Calculate the determinant, trace, square and inverse of each Pauli matrix. [4]

(ii) The anticommutator of two matrices $\boldsymbol{A}$ and $\mathbf{B}$ is defined as $[\mathbf{A}, \mathbf{B}]_+ = \mathbf{AB} + \mathbf{BA}$. Show that $[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j]_+ = 2\delta_{ij}\boldsymbol{I}$, where $\boldsymbol{I}$ is the 2×2 identity matrix. Show that $Tr[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j] = 2\delta_{ij}$. [4]

(iii) The matrix $\boldsymbol{\sigma}(\mathbf{a}) = a_x\boldsymbol{\sigma}_x + a_y\boldsymbol{\sigma}_y + a_z\boldsymbol{\sigma}_z$ is defined in terms of the vector $\mathbf{a} = (a_x, a_y, a_z)$. Prove that for general non-zero vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\boldsymbol{\sigma}(\mathbf{a})\boldsymbol{\sigma}(\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{I} + \mathrm{i}\boldsymbol{\sigma}(\mathbf{a} \times \mathbf{b}),$$

where $\mathbf{I}$ is the 2×2 unit matrix. Find a necessary and sufficient condition that the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ must satisfy in order that $\boldsymbol{\sigma}(\mathbf{a})\boldsymbol{\sigma}(\mathbf{b}) - \boldsymbol{\sigma}(\mathbf{b})\boldsymbol{\sigma}(\mathbf{a}) = 0$. [6]

12. Let L be the differential operator

$$L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$$

with constant coefficients a and b, and let u_1 and u_2 be solutions to the equation $L(y) = 0$. Define the function

$$W(x) \equiv u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}.$$

(a) Show that the inhomogeneous equation $L(y) = R(x)$ has a particular solution

$$y_1(x) = t_1(x)u_1(x) + t_2(x)u_2(x)$$

where $R(x)$ is an arbitrary function of x, and

$$\begin{aligned} t_1(x) &= -\int u_2(x)\frac{R(x)}{W(x)}dx \\ t_2(x) &= \int u_1(x)\frac{R(x)}{W(x)}dx, \end{aligned}$$

assuming $W(x) \neq 0$. [6]

(b) Prove that W satisfies the equation

$$\frac{dW}{dx} + aW = 0,$$

and that, as a result, W is either identically zero, or $W(x) \neq 0$ for all x. [4]

(c) Assume $u_1(x) \neq 0$ for all x. Show that if u_1 and u_2 are linearly dependent, then $W(x) = 0$. [3]

(d) Using the method of (a), or otherwise, find the general solution $y(x)$ to the equation

$$\frac{d^2y}{dx^2} + 4y = 1/\cos(2x)$$

on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [7]