A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 12 June 2019, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is permitted.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Write the following in the form $a + i\,b$ where a and b are real:

$$\text{(a) } (2\mathrm{e}^{i\pi/4})^2\,, \qquad \text{(b) } \frac{1+i}{1-i}\,, \qquad \text{(c) } \ln(\sqrt{3}+i)\,, \qquad \text{(d) } (i^i)^i\,.$$

[4]

2. Solve

$$\det\begin{pmatrix} z & -i \\ -i & z^3 \end{pmatrix} = 0\,,$$

and show your solutions on an Argand diagram. [4]

3. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-4}{x+y}\,.$$

[5]

4. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2\,,$$

[6]

for $y(1) = 1$.

5. A plane is defined as $3x + 4y + 12z = 26$. What is the smallest distance from the point (1,1,1) to the plane? [4]

6. Prove that the non-zero eigenvalues of an anti-Hermitian matrix, for which $\mathbf{A}^\dagger = -\mathbf{A}$, are purely imaginary, and that the eigenvectors corresponding to different eigenvalues are orthogonal. [5]

7. Find the eigenvalues and a set of three eigenvectors for the matrix

$$\begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}.$$

[6]

8. Prove the vector identity $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \det(\mathbf{a},\mathbf{b},\mathbf{d})\,\mathbf{c} - \det(\mathbf{a},\mathbf{b},\mathbf{c})\,\mathbf{d}$, where $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ are vectors in $\mathbb{R}^3$ and $(\mathbf{a},\mathbf{b},\mathbf{c})$ denotes a matrix where each column is given by the respective vector. [6]

Section B

9. (a) Show that

$$64\sin^7(\theta) = 35\sin(\theta) - 21\sin(3\theta) + 7\sin(5\theta) - \sin(7\theta).$$

[6]

(b) Show that

$$\cos^{-1}(z) = -i\ln\left(z \pm i\sqrt{1-z^2}\right),$$

and justify all steps in your working. [5]

(c) For the polynomial equation

$$z^5 - 5z^4 + 11z^3 - 3z^2 - 12z + 8 = 0,$$

find:

(i) the sum of the roots;

(ii) the product of the roots;

(iii) the values of all the roots.

(HINT: One of the roots is $(2+2i)$.) [9]

10. A mass attached to a spring of spring constant k, executing driven, damped, simple harmonic motion can be described by the following differential equation:

$$m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + m\gamma\frac{\mathrm{d}z}{\mathrm{d}t} + kz = F_0\cos(\omega t + \phi_0)\,.$$

(a) Under what conditions does the auxiliary equation (with $F_0 = 0$) show oscillating solutions? Write down an explicit form for these oscillating solutions. [3]

(b) Find a condition on the parameters m, γ, k, and ω such that the steady state solution has the form:

$$z(t) = A\cos(\omega t)\,,$$

and find a general expression for the amplitude A in terms of $\omega, k, m,$ and F_0. [6]

(c) What is the average power, $\bar{P}$, transferred? Show that the maximum value is

$$\bar{P}_{\max} = \frac{F_0^2}{2\,m\,\gamma}.$$

[6]

(d) Sketch a graph of the average power as a function of the angular frequency ω and mark the half-power points at ω_1 and ω_2. For a sharp resonance show that $|\omega_1 - \omega_2| \approx \gamma$. [5]

11. (a) The exponential of a matrix $\mathbf{A}$ is defined as:

$$\exp(\mathbf{A}) = \sum_{m=0}^{\infty} \frac{\mathbf{A}^m}{m!}.$$

Show that if $\mathbf{A}$ is Hermitian, then the exponential of $\mathbf{A}$ is also Hermitian. Find the matrix elements of the exponential of $\mathbf{A}$ in terms of its matrix elements for the case where $\mathbf{A}$ is a diagonal. [4]

(b) Find $\exp(\mathbf{A})$ for the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}.$$

[6]

(c) An idempotent matrix $\mathbf{M}$ is a matrix for which $\mathbf{M}^2 = \mathbf{M}$.

For a general idempotent matrix $\mathbf{M}$, show that:

(i) $\mathbf{M}$ must be a square matrix;
(ii) if $\mathbf{M}$ is invertible, it must be the identity matrix $\mathbf{I}$;
(iii) $\mathbf{I} - \mathbf{M}$ is also idempotent;
(iv) $\det(\mathbf{M}) = 0$ or 1;
(v) if $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal unit vectors, $\mathbf{M} = \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T$ is idempotent;
(vi) $\exp(\alpha\mathbf{M}) = \mathbf{I} + \mathbf{M}(\exp(\alpha) - 1)$, where α is a scalar. [7]

(d) Show that a necessary condition for a 2×2 matrix to be idempotent is that it is diagonal or that its trace equals 1. [3]

12. (a) Write the set of simultaneous equations

$$\begin{aligned} -3x_2 - 2x_3 &= 4 \\ x_1 - 4x_2 - 2x_3 &= 2 \\ -3x_1 + 4x_2 + x_3 &= -1 \end{aligned}$$

in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a matrix, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices, and solve for $\mathbf{x}$. [6]

(b) Solve the equation

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + \mathbf{Mx} = 0\,,$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^2$, and the matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}.$$

[6]

(c) Give the necessary and sufficient conditions for a matrix to be a rotation matrix, and use these to show that

$$\mathbf{R} = \begin{pmatrix} \frac{1}{3}(1+\sqrt{3}) & \frac{1}{3}(1-\sqrt{3}) & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3}(1+\sqrt{3}) & \frac{1}{3}(1-\sqrt{3}) \\ \frac{1}{3}(1-\sqrt{3}) & \frac{1}{3} & \frac{1}{3}(1+\sqrt{3}) \end{pmatrix}$$

is a rotation matrix.

Consider a triangle defined by the points $\mathbf{a} = 2\mathbf{i}\,,\ \mathbf{b} = 2\mathbf{j}$, and $\mathbf{c} = 2\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are an orthonormal set of basis vectors. What is the area of the triangle? Now calculate the positions of the rotated triangle defined by points $\mathbf{a}' = \mathbf{Ra}$, $\mathbf{b}' = \mathbf{Rb}$, and $\mathbf{c}' = \mathbf{Rc}$, respectively. What is the area of the rotated triangle? [8]