A10287W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP1: PHYSICS 1

also

Preliminary Examination in Physics and Philosophy

Monday 10 June 2019, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. A particle is projected from the ground inside a horizontal tunnel of height H. Neglecting air resistance, show that the maximum range inside the tunnel is

$$\sqrt{8H\left(\frac{u^2}{g}-2H\right)},$$

where u is the initial speed of the particle and g is the acceleration due to gravity. [6]

2. Consider a non-relativistic collision between two particles of masses M_1 and M_2. Show that the total kinetic energy in the laboratory frame can be written as

$$\frac{1}{2}(M_1+M_2)V_{\mathrm{CM}}^2+\frac{1}{2}\mu v^2\,,$$

where V_{CM} is the speed of the centre of momentum frame, v is the relative speed between M_1 and M_2, and μ is the reduced mass. State the physical meaning of the two terms in the expression, and deduce the loss of kinetic energy for a perfectly inelastic collision. [7]

3. (a) Show that the moment of inertia of a thin uniform rod of mass M and length L, about an axis passing through the rod's centre of mass and inclined at an angle α to its long axis, is given by

$$\frac{1}{12}ML^2\sin^2\alpha\,.$$

(b) Making use of the parallel axis theorem, or otherwise, find the moment of inertia of the rod about an axis in the plane of the rod, parallel to its long axis and a perpendicular distance D away. [5]

4. A particle of mass M is travelling at constant speed towards the rod from the previous question, in a direction perpendicular to the rod's long axis. The rod has the same mass M as the particle, and is initially at rest. Assuming a perfectly elastic collision, at what distance from the centre of the rod should the collision occur, so that the particle and centre of the rod move with equal speeds afterwards? Show that the relative speed of the particle and the point of contact on the rod is the same before and immediately after the collision. [7]

5. Write down an expression for the invariant interval in space-time. Using the form of this invariant, or otherwise, derive an expression for the time dilation between observers moving at constant relative speed.

A clock is placed in a jet aircraft. The clock measures a time interval of 1000 s when the aircraft flies directly from one location to another at a constant speed of $300\,\mathrm{m\,s^{-1}}$. According to special relativity, how much longer or shorter is the time interval for the aircraft's journey, as measured by clocks at rest on the ground? [7]

6. Derive the formula for the longitudinal relativistic Doppler effect. At what speed and in which direction would you have to move a laser source emitting light with wavelength $\lambda = 633\,\text{nm}$ in order to trigger an infrared sensor moving directly towards you at $0.2c$, whose response in its own rest frame is at 1000 nm? (Assume that all motion occurs along the same axis.) [8]

Section B

7. (a) A rocket of total instantaneous mass $M \equiv M(t)$ (including fuel) is launched vertically upwards from the surface of the Earth, and emits fuel backwards at a constant speed u relative to the rocket. Starting from consideration of the change in momentum of the rocket+fuel, derive the rocket equation

$$F = M\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}M}{\mathrm{d}t}\,,$$

where F is the net external force, and $v \equiv v(t)$ is the rocket's instantaneous velocity. [5]

The rocket moves under the influence of the Earth's gravity g, which may be assumed to be constant, and is also subject to a linear air resistance kMv. Given that fuel is burned at a rate such that $M(t) = M_0\mathrm{e}^{-bt}$, where M_0 is the initial total mass of the rocket, and b is a positive constant, find an expression for the velocity of the rocket, $v(t)$, at a time t. Give the condition on b in order for the rocket to launch. [5]

(b) Now consider two buckets filled with sand that are hanging on either side of an ideal, massless, frictionless pulley and connected by a massless rope. Initially, both buckets contain the same amount of sand and each has a total mass of M_0 (including the sand). However, one of the buckets has a small hole in it and is leaking, resulting in a loss of mass at a constant rate α.

Given that $M(t)$ and $v(t)$ are, respectively, the instantaneous mass and upwards velocity of the leaky bucket, show that

$$\frac{\mathrm{d}v}{\mathrm{d}M} = \frac{g}{\alpha}\left(\frac{M(t) - M_0}{M(t) + M_0}\right)\,.$$

Given that ϵM_0 $(0 \leq \epsilon < 1)$ is the initial mass of the sand in each bucket, and the buckets are initially at rest, determine the speed of the leaky bucket at the instant it becomes empty. (You may assume that the bucket will empty before it hits the pulley.) Evaluate your result for the cases $\epsilon = 0$ and $\epsilon \to 1$. [10]

8. (a) A point particle of mass m moves under the influence of a central force $F(r)$, where r is the distance of the particle from the force centre (taken to be the origin). Show that the energy of the particle can be written

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{J^2}{2mr^2} + U(r)\,,$$

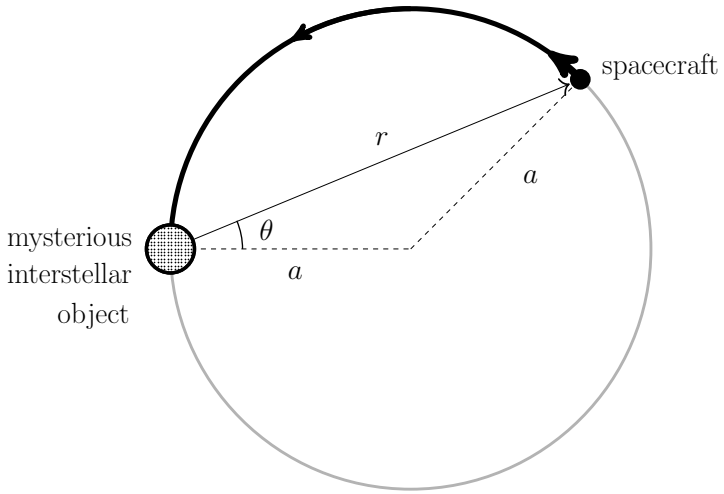
where J is the angular momentum about the origin. Hence, derive an expression for $m\left(\frac{\mathrm{d}^2r}{\mathrm{d}t^2}\right)$ and comment on its relation to Newton's second law. [6]

(b) Explain what is meant by the *effective potential*. Sketch the effective potential for the case of an attractive central force $F(r) = -kr^3$, where k is a positive constant, and use it to explain why all orbits are bounded. Find the radius and energy of any circular orbits, in terms of J, k and m. [7]

(c) The equation describing the orbit of an object moving under the influence of a general central force may be written as

$$F(1/u) = -\frac{J^2u^2}{m}\left(u + \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\right)\,,$$

where $u = 1/r$ and θ is the azimuthal angle in the plane of the orbit.

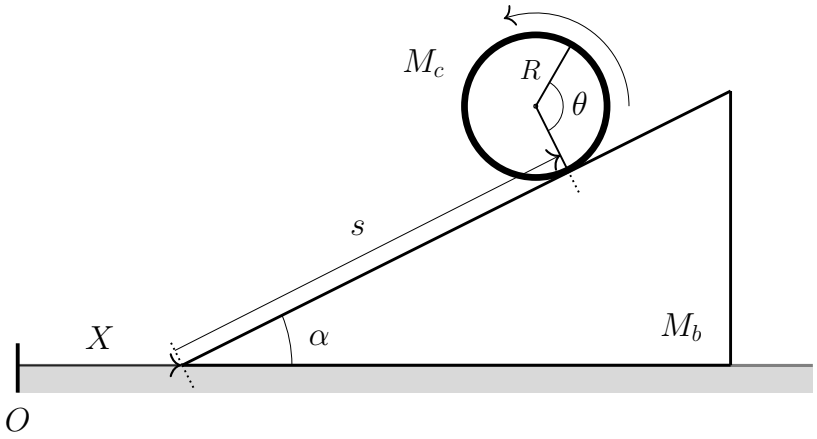


Suppose that a spacecraft close to a mysterious interstellar object finds itself caught on a circular orbit of radius a, as shown in the diagram. Show that there exists a central force $F(r) = -Cr^n$ responsible for this orbit, where r is the distance of the spacecraft from the object, and find an expression for C (in terms of J, m and a) and the value of n. [7]

9. (a) Write down the Euler-Lagrange equations for a system described by generalised coordinates q_i and generalised velocities $\dot{q}_i$.

Explain briefly what is meant by: (i) a cyclic coordinate, and its relation to a conservation law; and (ii) a holonomic constraint. [4]

(b) A uniform solid cylinder of mass M_c and radius R rolls without slipping down the inclined surface of a wedge-shaped block. The block has a mass M_b and an angle of inclination α, and is itself free to slide frictionlessly along a horizontal surface.



Justify that the system can be described in terms of two generalised coordinates, and find an expression for the Lagrangian in terms of $(X, s, \dot{X}, \dot{s})$, where X is the position of the lower left corner of the block and s is the distance along the inclined surface to the bottom of the cylinder.

Identify the cyclic coordinate and associated conserved quantity.

Use the Euler-Lagrange equations to determine the linear accelerations $\ddot{X}$ and $\ddot{s}$, and evaluate your expressions in the limit $M_b \gg M_c$. [9]

(c) Consider now the case where the block is fixed in position, and where the forces constraining the cylinder's motion (i.e. that it maintains contact with the block and rolls without slipping) are to be computed. Construct a Lagrangian for this problem, and use the Euler-Lagrange equations to determine the constraint forces, stating their physical origin. [Hint: introduce a new generalised coordinate, r, representing the perpendicular distance of the bottom of the cylinder from the inclined surface, and treat the system as having three generalised coordinates (r, s, θ), together with appropriate constraints.]

Given a coefficient of static friction μ_s, show that the cylinder rolls without slipping provided that $\mu_s \geq \frac{1}{3}\tan\alpha$. [7]

[The moment of inertia of the cylinder about its principal axis is $\frac{1}{2}M_cR^2$.]

10. (a) Given that the energy-momentum $(E/c, \mathbf{p})$ and space-time $(ct, \mathbf{r})$ 4-vectors transform in an identical way, write down the Lorentz transforms for energy and momentum for two inertial frames moving at relative speed v, along the x-axis.

Consider a group of N particles, where the ith particle has energy E_i in the laboratory frame. Starting from the Lorentz transforms show that the Lorentz factor of the centre of momentum frame, $\gamma_{\rm CM}$, can be written

$$\gamma_{\rm CM} = \frac{\sum_i^N E_i}{E_{\rm CM}} \,,$$

where $E_{\rm CM}$ is the total energy in the centre of momentum frame. Write down a relationship between $\gamma_{\rm CM}$ and the speed of the centre of momentum frame, $v_{\rm CM}$. [6]

(b) Consider a photon which collides with a proton at rest in the laboratory frame. The collision produces a neutral π meson and a proton via

$$\gamma + p \rightarrow \pi + p \,.$$

Show that the threshold (i.e. minimum) photon energy for this reaction to occur is

$$E_\gamma^{\rm th} = m_\pi c^2 \left(1 + \frac{m_\pi}{2m_p}\right) \,,$$

where m_p and m_π are the rest masses of the proton and π meson, respectively. [6]

(c) A π meson is produced at threshold via the interaction given in (b), and subsequently decays to two photons. One photon travels in a direction parallel, and the other anti-parallel, to the initial line of flight of the π meson in the laboratory frame. Calculate the energy of each photon in the laboratory frame. [8]

[The rest mass energy of the proton is 938 MeV and of the π meson is 135 MeV.]