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FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SECOND PUBLIC EXAMINATION

Honour School of Physics,
Parts A and B: 3 and 4 year Courses

SHORT OPTIONS

TRINITY TERM 2018

Tuesday, 12 June
9.30 am to 11.00 am for candidates offering ONE Short Option
9.30 am to 12.30 pm for candidates offering TWO Short Options

Answer **two** *questions from each option for which you have entered.*

Start the answer to each question in a fresh book.

The use of approved calculators is permitted.

If you have entered for two Short Options,
keep your answers to the two options in different books
and at the end hand in two bundles, one for each option.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

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Some sections may start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. Define the concepts of a branch point and a branch cut, illustrating your answer using the function $\sqrt{z}$. [4]

Show that

$$f(z) = \sqrt{\frac{z-1}{z+1}}$$

is holomorphic on the complex plane cut on the real line segment $[-1, +1]$. [4]

Find the first two terms in the expansion of $f(z)$ around $z = \infty$ and hence calculate

$$I_\Gamma = \oint_\Gamma f(z)\,\mathrm{d}z$$

where the contour Γ encloses the real line segment $[-1, +1]$. By collapsing Γ onto the cut use this result to show that

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = \pi$$

[8]

By choosing a suitable generalization of $f(z)$ use the same method to find

$$\int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{1/n} \mathrm{d}x$$

where n is an integer ≥ 2. [9]

2. Using the methods of contour integration:

(a) Compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$$

and

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x$$

where $\mathcal{P}$ denotes principal value. [15]

(b) Compute

$$\int_0^{2\pi} \frac{1}{13 + 12\sin\theta}\,\mathrm{d}\theta$$

[10]

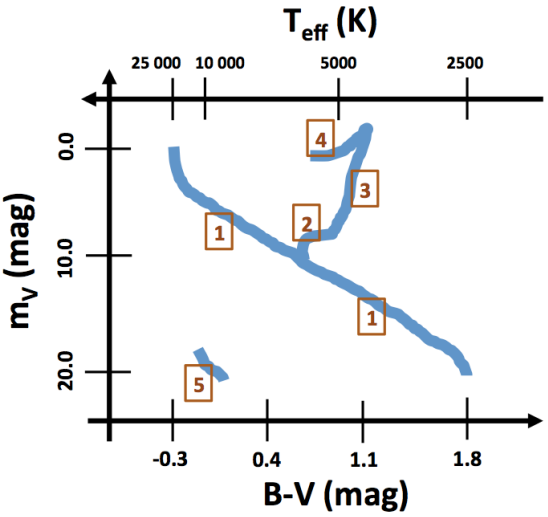
3. Show that the map $z \to w = f(z)$, where $f(z)$ is holomorphic, is locally angle preserving (conformal) except at those points for which $f'(z) = 0$. Find how the map affects angles at z_0 if $f'(z_0) = 0$ but $f''(z_0) \neq 0$. [8]

Show that for $f(z) = \sin \pi z/2$ the semi-infinite strip $-1 \leq \mathrm{Re}\, z \leq 1,\ \mathrm{Im}\, z \geq 0$ is mapped to the w upper half plane. What is the image in the w plane of the line segment $-1 \leq \mathrm{Re}\, z \leq 1,\ \mathrm{Im}\, z = a$, where $a > 0$ is a constant? [10]

Find the inverse map $f^{-1}(w)$ and show that it is not holomorphic at those points where the original map is not conformal. By considering the behaviour of angles under the map, explain why this is necessarily the case. [7]

Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. Consider the schematic colour-magnitude diagram below, of a star cluster located at a distance of 100 pc, where star formation has been uncharacteristically continuous.



(a) Determine the range of radii (in Solar radius $R_\odot$) for cluster stars along the main sequence. [8]

(b) Using the equation of hydrostatic equilibrium and the ideal gas law, derive two estimates of the central pressure of stars, and thus derive the mass-radius relation for stars along the main sequence. What is the range of masses (in Solar mass $M_\odot$) for cluster stars along the main sequence? Based on the colour-magnitude diagram, is this range consistent with the usual main sequence mass-luminosity relation $L \propto M^4$? [9]

(c) Very massive stars are actually dominated by radiation pressure. What is the mass-radius relationship for those stars? [3]

(d) Name and briefly describe the stellar evolution stages identified by the numbers 1 to 5 in the colour-magnitude diagram. [5]

[*Assume* $\mathrm{M}_{V,\odot} = 5.0$. *Ideal gas law:* $P = \frac{\rho}{\mu\, m_\mathrm{H}} k_\mathrm{B}\, T$, *where* P *is the pressure,* ρ *the density,* μ *the mean molecular mass,* m_H *the mass of the hydrogen atom,* k_B *the Boltzman constant, and* T *the temperature. Radiation pressure:* $P = \frac{1}{3}\, a\, T^4$, *where* P *is the pressure,* a *is a constant, and* T *the temperature.*]

2. (a) A nearby disk galaxy has a redshift $z = 0.005$ and stellar populations similar to the Sun. The rotation of its stars is observed to be flat with radius at a value of $V_{\rm rot} = 315\,{\rm km\,s^{-1}}$, all the way to the optical edge of the galaxy at a radius $R_{\rm out} = 220''$. Estimate the luminous and total masses of this galaxy and discuss the need for dark matter. [9]

(b) The effective (half-light) radius of this galaxy is measured to be $R_{\rm eff} = 22''$ in the V band. Discuss the need for dark matter in the inner parts of the galaxy. What type of galaxy is this galaxy likely to be? [4]

(c) A nearly identical galaxy is observed in a distant cluster of galaxies, with a galaxy systemic velocity of $6500\,{\rm km\,s^{-1}}$ and an apparent magnitude $\mathrm{m}_V = 12.5\,{\rm mag}$. Discuss the likely position and movement of this galaxy in the cluster. [5]

(d) Derive the Tully–Fisher relation for disk galaxies from basic principles, and show that the slope of the relation is naively expected to be -10 rather than -8 as observed. How should key properties of galaxies vary as a function of their rotation velocities to explain the difference? [7]

[*Assume* $\mathrm{M}_{V,\odot} = 5.0$, *and* $H_0 = 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$. *The observed Tully–Fisher relation for disk galaxies is*

$$\mathrm{M}_V = -8\,[\,\log(V_{\rm rot}/{\rm km\,s^{-1}}) - 2.5\,] - 22.5,$$

where M_V *is the absolute magnitude of a galaxy in the* V *band and* $V_{\rm rot}$ *its rotation velocity.*]

3. (a) The Friedmann–Robertson–Walker (FRW) equation is

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_{\rm M}}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}\,,$$

where R is the scale factor of the Universe, $\dot{R}$ is its derivative with respect to time, G is the gravitational constant, k is a constant, c is the speed of light, and Λ is the cosmological constant. Here we assume that $\rho_{\rm M}$ consists solely of the mass density of non-relativistic matter in the Universe. What is the physical meaning of k? Discuss its possible values. [4]

(b) If we define the Hubble parameter $H \equiv \dot{R}/R$, and take H_0 to be its value today, show that the FRW equation can be written as

$$H^2 = H_0^2\left(\frac{\Omega_{\rm M}}{R^3} + \frac{\Omega_k}{R^2} + \Omega_\Lambda\right)\,, \qquad (1)$$

where $\Omega_{\rm M}$, Ω_k, and Ω_Λ are constants. Find expressions for $\Omega_{\rm M}$, Ω_k, and Ω_Λ. [5]

(c) Let us assume that the Universe has no cosmological constant. First, find a solution for R as a function of time if $\Omega_k = 0$ and $\Omega_{\rm M} \neq 0$. Estimate the present age of the Universe (express your answer in terms of H_0). Second, find a solution for R as a function of time if $\Omega_{\rm M} = 0$ and $\Omega_k \neq 0$, and again estimate the present age of the Universe (in terms of H_0). Explain why the Universe is older in one case than in the other. Illustrate your explanation with a diagram. [10]

(d) Let us assume that we live in a dust-dominated Universe, that the cosmic microwave background has a current temperature of approximately 3 K, and that hydrogen in a gas ionises at a temperature approximately 3000 K. Estimate the redshift and age of the Universe (in years) at which recombination took place. What is the maximum distance light could have travelled in that time? What is the physical size of that distance today and what does it represent? Discuss briefly. [6]

[*Use* $H_0 = 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$.]

Section S3 QUANTUM IDEAS

1. (a) Explain the de Broglie Hypothesis. [5]

(b) One can estimate the conditions under which quantum effects become important by comparing the de Broglie wavelength of the system under consideration to its physical extent. In thermal equilibrium, at temperature T, the average kinetic energy of a system is given by $p^2/2M = k_{\mathrm{B}}T/2$, where p is the momentum, M is the mass, and k_{B} is the Boltzmann constant. Derive the de Broglie wavelength of this system as a function of temperature. [4]

(c) The lattice spacing between atoms in a typical solid is $d = 0.3\,\mathrm{nm}$. Find the temperature below which the free electrons in a solid are likely to behave quantum mechanically. [6]

(d) What can you say about the de Broglie wavelength of the atomic nuclei at the same temperature? [5]

(e) What is, therefore, the appropriate way of treating a solid, assuming that it is a collection of nuclei and electrons, at room temperature? What degrees of freedom ought to be quantized and what can remain classical? Would the same be true if the solid were to be cooled down to 1 K? [5]

2. The state of a particle is represented by the spatial wave function $\Psi(x) = A(a^2 - x^2)$ for $-a \leq x \leq a$ and is zero outside of this region.

(a) Determine the constant A from the normalisation condition. [5]

(b) Calculate the expectation values of x and x^2. [6]

(c) Calculate the expectation values of the momentum p and of p^2. [6]

(d) Check that the uncertainty principle holds. [4]

(e) Matter waves are diffracted through a slit of width d and the interference fringes are observed in the far field. What would happen to the width of the central maximum if the size of the slit is halved, i.e. if d was reduced by a factor of 2? [4]

3. (a) Describe the phenomenon of quantum tunneling. [4]

(b) The probability of quantum tunneling through a barrier of height V and width L is roughly given by $P = \mathrm{e}^{-2\alpha L}$. Derive the value of α in terms of the particle's mass, its energy E and the height of the potential V. [6]

Heavier elements are generated inside the Sun through nuclear fusion leading to the binding of atoms of the lighter elements, such as hydrogen and helium, into heavier ones, thereby releasing the energy that powers the Sun. The temperature in the centre of the Sun is about 10 million degrees Kelvin. The average energy of the protons at temperature T is approximately $k_{\mathrm{B}}T$ where k_{B} is the Boltzmann constant.

(c) Make an estimate of the probability, based on quantum tunneling, that two protons, having come to within 1 fm of one another, will overcome their Coulomb repulsion and bind together. [9]

(d) In reality, this probability is actually about 10^{-28}. Name two possible factors that could account for the large discrepancy between your estimate and the actual value. Why is the actual small probability still sufficient to power the Sun? [6]

Section S4 ENERGY STUDIES

1. Describe and discuss the fundamental assumptions, and the associated loss mechanisms, used to derive the Shockley–Queisser (SQ) limit for the conversion of sunlight into electricity using solar cells. [7]

Name three solar cell device concepts that exist or are currently being investigated to overcome the SQ limit, and describe the loss processes each of them addresses. [6]

In Oxford (latitude 51.75° North), a $5\,\mathrm{kW_p}$ solar module array is installed with a tilt angle of 41.1° (where 0° tilt angle is horizontal) facing south towards the equator. Assuming an intensity of $1\,\mathrm{kW\,m^{-2}}$ of solar radiation on a horizontal surface in Oxford at the equinox months (March and September), give an estimate for the maximum power output of this solar module array for the summer and winter solstice months (June and December). [5]

Compare this to the array's yearly generated electricity of 4.1 MWh and comment. [3]

Assuming we covered 5% of the United Kingdom with 20% efficient solar modules, make a reasoned estimate of how much energy per day per person this would result in and how this would compare to the average energy use per person. [4]

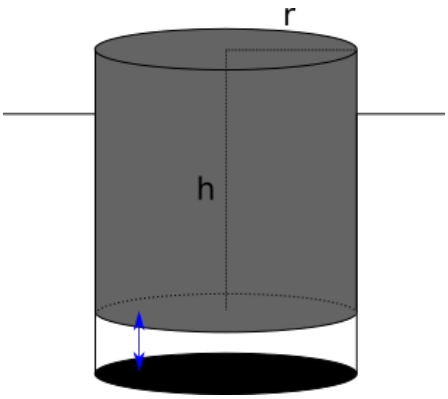
2. Sketch the main components of a pumped storage hydroelectric plant and describe its operational principle. [4]

The world's largest pumped storage plant has a head of 400 m and an upper reservoir of 14,000,000 m^3. The pumped storage plant's maximum generation capacity is rated at 3 GW_e. How much usable energy can be stored in this "battery"? Calculate the flow rate of water, and give an estimate of the number of hours the plant could operate at full power. Assume the efficiency of generation is 89% and the density of water is 1 g cm^{-3}. [5]

Estimate the main dimensions of a pumped storage hydroelectric plant that would contain the same amount of energy as one litre of gasoline ($\approx$ 9 kWh) and comment. [4]

Describe advantages and disadvantages of pumped storage hydroelectric plants. [6]

One currently developed approach for pumped hydroelectric storage is based on hydraulic lifting of a large cylinder of rock using water pumps (see figure).



The cylinder of rock gains potential energy when lifted, which can be converted back when the water under the cylinder is released back through a turbine. Assuming a density of 3,000 kg m^{-3} for the rock (neglect buoyancy), a cylinder of radius r and height $h = 2r$, a maximum lift height of r and an efficiency of 89%, estimate r to match the storage capacity of the pumped hydroelectric storage of the first part of this question. [4]

Discuss how you expect the cost of such storage to scale with the radius r. [2]

3. Comment and discuss *critically* with numerical estimates, where appropriate, on **three** of the following topics or statements: [25]

(a) Describe and compare the two main approaches for harvesting energy from nuclear reactions.

(b) Large scale energy infrastructure is a thing of the past.

(c) Compare the land use of five different energy conversion technologies and the implications for their deployment.

(d) Compare three different energy storage technologies based on different physical principles.

(e) How useful is the concept of learning curves for nuclear fission reactors, wind power and photovoltaics in predicting successful technologies?

(f) Assuming capital cost dominate the cost for backup power, estimate the extra charge per kWh of electricity in the UK if 100% backup power were provided here by gas powered stations. Discuss the implications.

(g) Compare the land use area needed to produce the energy for driving a car a distance of 20,000 km per year using biofuels and solar in the UK.

Section S7 CLASSICAL MECHANICS

1. Consider the motion of a particle with positive charge q and mass m in a simple model field that approximates the field in the "tail" of the earth's magnetic field. Specifically the magnetic vector potential is $\mathbf{A} = -\frac{1}{2L}B_0 x^2 \hat{\mathbf{z}}$ where B_0 and L are positive constants, and the electrostatic potential is zero.

(a) Obtain an expression for the magnetic field. [2]

(b) Write down the Lagrangian for the particle. [2]

(c) Obtain expressions for the three canonical momenta, p_x, p_y and p_z. Which of these momenta are constants of the motion? [6]

(d) Obtain an expression for the Hamiltonian. Is the Hamiltonian a constant of the motion? [4]

(e) Consider the initial conditions at time $t = 0$ where the velocity $v_y = \mathrm{d}y/\mathrm{d}t = 0$, $p_z = p_{z0}$, $H = H_0$ and $x = x_0$. Obtain expressions for $\dot{x}$ and $\dot{z}$ as functions of x and the initial conditions. [3]

(f) For the same initial conditions as part (e) give an expression for $\mathrm{d}x/\mathrm{d}z$ and sketch the trajectory in the xz plane for the cases (i) $p_{z0} = 0$ and (ii) $p_{z0} = -\sqrt{2mH_0}$. To get full marks you must label the direction of travel along the trajectory, the maximum and minimum values of x, and the points where $\dot{z} = 0$. [8]

2. An axisymmetric top has the Lagrangian

$$L = \tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mga\cos\theta$$

where θ, ϕ, ψ are the usual Euler angles.

(a) Give expressions for p_θ, p_ϕ and p_ψ. Which of these canonical momenta are conserved? What other quantity is conserved? [7]

The top's Hamiltonian is (there is no need to derive this):

$$H = \frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + mga\cos\theta.$$

Consider a top that is spun so that the initial conditions are $\theta = 0$, $\dot{\phi} + \dot{\psi} = \omega$ and $p_\theta = I_1\omega_0$.

(b) Solve for $p_\phi(t)$, $p_\psi(t)$ and $H(t)$. Hence or otherwise show that

$$\omega_0^2 = \dot{\theta}^2 + \frac{I_3^2}{I_1^2}\omega^2\sin^2(\theta/2)\left[\frac{1}{\cos^2(\theta/2)} - \alpha\right] = \dot{\theta}^2 + V(\theta)$$

with

$$\alpha = \frac{4mgaI_1}{I_3^2\omega^2}$$

[7]

(c) Derive an expression for $\dot{\phi}$ in terms of θ, ω and the moments of inertia. [3]

(d) Plot $V(\theta)$ for $0 < \theta < \pi$ for (i) $\alpha < 1$ and (ii) $\alpha > 1$, identifying any minima. Describe the motion of the top's axis in each case when $\omega_0 \ll (I_3/I_1)\omega$, giving an approximate expression for the maximum θ in each case. Calculate the oscillation frequency in case (i). Give an expression for the period in case (ii). What happens to this period as $\omega_0 \to 0$? [8]

3. We define the poisson bracket of two functions $F(\mathbf{q}, \mathbf{p})$ and $G(\mathbf{q}, \mathbf{p})$ as:

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right) = \left(\frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial G}{\partial \mathbf{q}} \cdot \frac{\partial F}{\partial \mathbf{p}} \right)$$

We consider a mechanical system with a Hamiltonian $H(\mathbf{q}, \mathbf{p})$

(a) Write down Hamilton's equations for the evolution of $\mathbf{q}(t)$ and $\mathbf{p}(t)$. [3]

(b) Show that if $[F, H] = 0$ then $F(\mathbf{q}, \mathbf{p})$ is a constant of the motion. [5]

(c) In spherical coordinates the Hamiltonian for a particle in a central potential is

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V(r).$$

Derive an expressions for $\dot{r}$ and $\dot{\phi}$ and show that p_ϕ and $J^2 = p_\theta^2 + p_\phi^2/(\sin^2 \theta)$ are constants of the motion. [7]

(d) Consider two dimensional motion on the equatorial plane ($\theta = \pi/2$ and $p_\theta = 0$) with a Newtonian gravitational potential $V = -k/r$. Show that

$$G = -\frac{p_r p_\phi}{m} \cos \phi + \frac{p_\phi^2}{mr} \sin \phi - k \sin \phi$$

is also a constant of the (two dimensional r, ϕ) motion. [4]

(e) We choose the origin of our angle ϕ so that $p_r = 0$ at $\phi = 0$ and therefore $G = 0$. Hence show that

$$\frac{1}{r} = \frac{mk}{p_\phi^2} + e \cos \phi$$

where e is an arbitrary ellipticity. You may wish to use the identity $\dot{r} = \dot{\phi} \frac{\mathrm{d}r}{\mathrm{d}\phi}$ and the results from part (c). [6]

Section S14 HISTORY OF PHYSICS

1. Assess the respective contributions to the Scientific Revolution of new instruments and the mathematization of nature.

2. What evidence was amassed during the eighteenth century to support the Newtonian system?

3. What contributions did Michael Faraday make to the science of electromagnetism?

4. To what extent was the discovery of the law of the conservation of energy an example of simultaneous discovery?

5. Why did "Big Science" emerge in the US when it did?

Section S16 PLASMA PHYSICS

1. Stating any assumptions that you make, show that when atomic hydrogen is heated in thermodynamic equilibrium, the free-electron number density, n_e, and the number density of neutral hydrogen atoms, n_H, satisfy the Saha equation

$$\frac{n_e^2}{n_H} = \frac{(2\pi m_e k_B T)^{3/2}}{h^3} \exp(-R/k_B T),$$

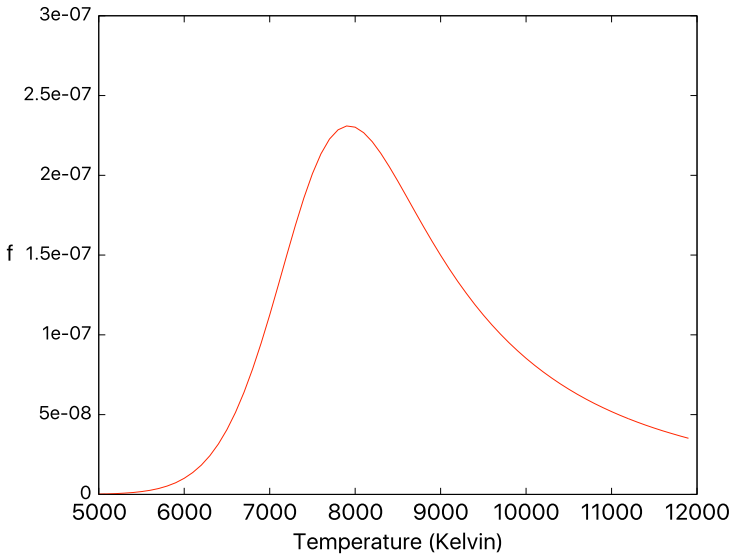
where R is the ionisation potential of atomic hydrogen (the Rydberg) and ideal gas partition functions have been used. [10]

Without calculations, explain briefly why the excited states of hydrogen can be ignored when calculating the partition function associated with the un-ionized hydrogen atoms. [3]

Consider a gas of hydrogen atoms that, when sufficiently cold to have negligible ionisation, has a number density of hydrogen atoms of n_0. Show that when heated, the ratio $f = n_2/n_0$ where n_2 is the number density of hydrogen atoms with an electron excited to states with principal quantum number 2, is given by

$$f = \frac{n_2}{n_0} \approx \frac{4}{(1 + n_e/n_H)} \exp\left(-\frac{3R}{4k_B T}\right).$$ [5]

The figure below shows the ratio f as a function of temperature as the hydrogen gas is heated.



Explain the qualitative form of the curve, calculate the degree of ionization (n_e/n_0) when f is at a maximum, and determine n_0. [7]

[A quantum state in hydrogen with principal quantum number n is bound with an energy of $-R/n^2$, and has a degeneracy of $2n^2$]

2. What is meant by the *plasma parameter*, N_D? Show that in a plasma with electron number density, n_e, and temperature, T, it is given by

$$N_\mathrm{D} = \frac{4\pi}{3} n_\mathrm{e} \left(\frac{\varepsilon_0 k_\mathrm{B} T}{n_\mathrm{e} e^2} \right)^{3/2},$$

and that the natural frequency of oscillation of the electrons in the plasma, ω_p, is

$$\omega_\mathrm{p}^2 = \frac{n_\mathrm{e} e^2}{\varepsilon_0 m_\mathrm{e}}.$$

[8]

The thermal conductivity, K, of a gas or plasma is related to the the heat capacity per unit volume, C, the mean velocity of the particles, v, and the collision time, τ, by

$$K = \frac{1}{3} C v^2 \tau.$$

Assuming that a reasonable approximation of the collision time of an electron in a plasma with ions of charge Z is given by $\tau \approx N_\mathrm{D}/Z\omega_\mathrm{p}$, show that for a plasma

$$K = \frac{B}{Z\sqrt{m_\mathrm{e}}} \left(\frac{2\pi k_\mathrm{B} \varepsilon_0^2}{3e^4} \right) (k_\mathrm{B} T)^{5/2}.$$

[5]

where B is a numerical factor that need not be evaluated, and takes into account the weighting of the velocity moments in the Maxwell–Boltzmann distribution.

For a plasma with a temperature gradient, $\mathrm{d}T/\mathrm{d}x$, the heat flow per unit area, $\dot{q}$, is given by Fourier's law of conduction, $\dot{q} = -K(\mathrm{d}T/\mathrm{d}x)$. However, this law will start to break down if the temperature gradient is too high, because there is a maximum physically possible heat flux, $\dot{q}_\mathrm{max}$, corresponding to all of the thermal energy being transported at the thermal velocity. Justify why this maximum heat flux is given by

$$\dot{q}_\mathrm{max} = \frac{3 n_\mathrm{e} k_\mathrm{B} T}{2} \sqrt{\frac{k_\mathrm{B} T}{m_\mathrm{e}}},$$

and show that if Fourier's law breaks down when $\dot{q} > A\dot{q}_\mathrm{max}$ (where $A < 1$) then this occurs for temperature gradients such that

$$\left(-\frac{\mathrm{d}T}{\mathrm{d}x} \right) > \left(\frac{9AZ}{B 4\pi k_\mathrm{B} T} \right) \left(\frac{n_\mathrm{e} e^4}{k_\mathrm{B} \varepsilon_0^2} \right).$$

[8]

In a laser-produced carbon ($Z = 6$) plasma, with an electron density of $10^{27}\,\mathrm{m}^{-3}$ and at temperatures of $10^7\,\mathrm{K}$, the temperature gradient between the critical and ablation surface can often exceed $10^{11}\,\mathrm{K\,m}^{-1}$. Given that $B \sim 10$, what would the minimum value of A need to be for Fourier's law to be invalid? [4]

3. In a uniformly magnetized plasma the eigenmodes for electromagnetic waves for waves travelling parallel to the magnetic field correspond to left and right-handed circularly polarized light, and the refractive index μ is given by

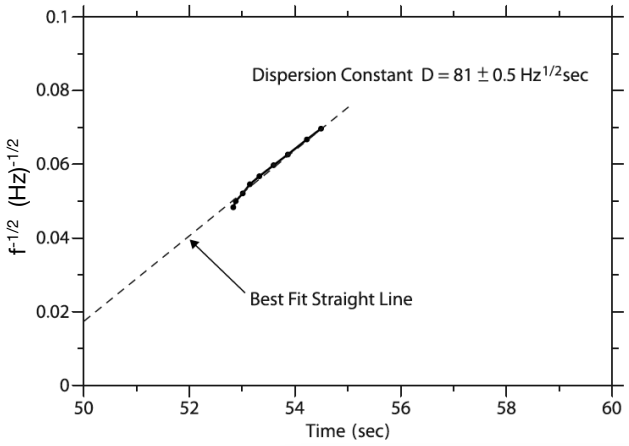
$$\frac{c^2k^2}{\omega^2} = \mu_{\mp}^2 = 1 - \frac{\omega_{\rm pe}^2}{\omega(\omega \mp \omega_{\rm ce})}\,,$$

where the minus sign corresponds to the right handed wave, and the plus sign to the left handed wave, $\omega_{\rm pe}$ is the electron plasma frequency, and $\omega_{\rm ce}$ the electron cyclotron frequency. Show that there is a frequency below which left handed waves cannot propagate, whereas for right handed waves there is a gap in frequency-space within which propagation is not possible. [6]

Show that for extremely low frequency right handed waves travelling through a uniform slab of magnetised plasma of thickness L, the arrival time, t, of the waves of frequency $f = \omega/2\pi$, varies such that they are dispersed according to

$$\frac{{\rm d}(f^{-1/2})}{{\rm d}t} = \frac{2c\sqrt{2\pi\omega_{\rm ce}}}{L\omega_{\rm pe}}.$$ [7]

During its mission to Saturn, when close to its equatorial plane, the Cassini probe recorded whistler waves caused by a lightning strike close to the northern pole of the planet. The plasma surrounding Saturn is not uniform, but forms a thick region in the equatorial plane. At the time of the recording, it is estimated that the waves travelled through a region of the plasma of a thickness approximately equal to the radius of Saturn (60,000 km). Magnetometers on the space craft showed that the magnetic field in this region was approximately 80 nT. The figure below shows the inverse of the square root of the frequency of the waves as a function of their arrival time. Estimate the mean electron density in the plasma surrounding the Cassini probe at the time of measurement. [8]



In this region around Saturn, how wide is the gap in frequency space where right handed waves cannot propagate? [4]

Section S18 ADVANCED QUANTUM MECHANICS

1. (a) A non-relativistic quantum particle of mass m is incident on the one-dimensional potential $U(x)$, where $U(x) \to 0$ for $|x| \to \infty$. Set up a scattering problem and write down the asymptotics of the wave function ψ_L (ψ_R) at $x \to \pm\infty$, assuming that the particle is incident on the potential from the left (right). Show that the reflection and transmission coefficients for a given potential depend only on the particle's energy and not on the direction from which it is incident on the potential. [4]

[*Hint: Multiply the Schrödinger equation obeyed by ψ_L by ψ_R and the equation obeyed by ψ_R by ψ_L, subtract and integrate over the real line. Compute the resulting quantity using the asymptotics of $\psi_{L,R}$.*]

(b) Using Fourier transform or any other method, show that the Green's function $G(x,x')$ obeying the equation $\hat{L}\,G(x,x') = \delta(x-x')$ and the boundary conditions $G(x,x') \to C\exp(\mathrm{i}k|x|)$ for $|x| \to \infty$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - E$ is the one-dimensional Schrödinger operator for a free particle with $E > 0$, C is a constant and $k = \sqrt{2mE}/\hbar$, is given by $G(x,x') = \frac{\mathrm{i}m}{k\hbar^2}\exp(\mathrm{i}k|x-x'|)$. [5]

(c) For a particle incident from the left on the one-dimensional potential $U(x) \leq 0$, where $U(x) \to 0$ for $|x| \to \infty$, show that the wave function of a stationary scattering state of the particle satisfies the integral equation

$$\psi_s(x) = \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{i}m}{k\hbar^2}\int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k|x-x'|}U(x')\psi_s(x')\,\mathrm{d}x'\,.$$

Find the solution of the integral equation for the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. Define and find the transmission and reflection coefficients T and R, and show that $R + T = 1$. [6]

(d) Consider scattering of "slow" particles with energies obeying $ka \ll 1$ by a generic potential $U(x)$ characterised by typical strength U_0 and width a. Assume that the potential is "weak", i.e. that $U_0 \ll \hbar^2/ma^2$.

(i) Set up a scattering problem and show, by analysing the Schrödinger equation and matching its solution to the asymptotics at $x \to \pm\infty$, that to leading order in $ka \ll 1$ and $U_0 \ll \hbar^2/ma^2$, the solution in the region $|x| \leq a$ is well approximated by $\psi \approx C = const.$ [3]

(ii) By considering the integral equation for ψ in the limit $ka \ll 1$, show that $C = (1 + \frac{\mathrm{i}m\gamma}{k\hbar^2})^{-1}$. Relate γ and $U(x)$. Compare with the δ-function potential above. [3]

(iii) Find the locations of singularities of the transmission amplitude in the complex plane of k and E. [2]

(iv) Using the scattering data found above, identify the bound states in the potential $U(x) < 0$ and compute their energies to leading order in $U_0 \ll \hbar^2/ma^2$. What states do we expect to appear if $U(x) > 0$? [2]

2. (a) Show that the Green's function $G(x,x')$ obeying $\hat{L}\,G(x,x') = \delta(x-x')$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x) - E$ is the Schrödinger operator, for a particle in an infinite potential well of width a,

$$U(x) = \begin{cases} 0, & |x| < \frac{a}{2}\,, \\ \infty, & |x| > \frac{a}{2}\,, \end{cases}$$

is given by

$$G(x,x') = -\frac{2m}{k\hbar^2 \sin ka}\,\sin\left[\frac{k}{2}\left(x + x' - |x'-x| + a\right)\right]\cdot\sin\left[\frac{k}{2}\left(x + x' + |x'-x| - a\right)\right], \quad (*)$$

where $k^2 = 2mE/\hbar^2$. [You may use the identity $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$.]

Identify the singularities of the Green's function and show that they correspond to the energy levels E_n of the particle in the potential well. [7]

(b) Let $\{\varphi_n(x)\}$ be an orthonormal set of eigenfunctions of the Schrödinger operator $\hat{L}$ with eigenvalues $\lambda_n = E_n - E$. Argue that the Green's function can be expanded as $G(x,x') = \sum\limits_n c_n\varphi_n(x)$ and show that the coefficients c_n are given by $c_n = \varphi_n^*(x')/\lambda_n$. Thus show that the Green's function can be written as

$$G(x,x') = \sum_n \frac{\varphi_n^*(x')\varphi_n(x)}{E_n - E}\,. \qquad (**)$$

Fix $x > x'$ (or $x < x'$), compute the limits $\lim\limits_{E\to E_n}(E_n - E)G(x,x')$, where $G(x,x')$ is given by Eq. (*) and Eq. (**), correspondingly, and find the expression for $\varphi_n^*(x')\varphi_n(x)$ by comparing the two results. [7]

(c) Find the explicit form of $\varphi_n(x)$. Compare the result with the normalised wavefunctions of the particle in an infinite potential well. [2]

(d) Now consider the Schrödinger operator with a non-local potential, whose action on the wave-function ψ is given by $U(x)\psi(x) = \int\limits_{-\infty}^{\infty}\mathrm{d}x' V(x,x')\psi(x')$. Using the Green's function $G(x,x') = \frac{m}{\kappa\hbar^2}\exp(-\kappa|x-x'|)$ of a free particle with $E < 0$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the Schrödinger equation $\hat{L}\psi = 0$ with such a potential can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\kappa|x-x'|}V(x',x'')\psi(x'')\,\mathrm{d}x'\,\mathrm{d}x''\,.$$

Assume that the potential $V(x,x')$ is separable, i.e. $V(x,x') = -\alpha F(x)F^*(x')$, where F is a smooth function. Show that the solution of the Schrödinger equation has the form

$$\psi(x) = \frac{\alpha m C}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\kappa|x-x'|}F(x')\,\mathrm{d}x'\,,$$

where C is a constant. Write an explicit expression for C. [5]

(e) Show that the energy levels are determined by the equation

$$\kappa = \frac{\alpha m}{\hbar^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}F(x)F^*(x')\,\mathrm{e}^{-\kappa|x-x'|}\,\mathrm{d}x\,\mathrm{d}x'\,.$$

[4]

3. (a) The Dirac equation for a free relativistic particle of mass m and spin $s = 1/2$ in $D = 3 + 1$ dimensions can be written in the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{\mathrm{i}}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H_D\psi\,,$$

where $\alpha_i^2 = I$, $\beta^2 = I$, $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$, $\alpha_i\beta + \beta\alpha_i = 0$ and I is the identity matrix. (i) Show that each component ψ_α, $\alpha = 1, ...N$, of the Dirac spinor ψ satisfies the Klein–Gordon equation. (ii) Show that the eigenvalues of α_i and β are equal to ± 1. (iii) Show that $\mathrm{Tr}\,\alpha_i = 0$ and $\mathrm{Tr}\,\beta = 0$. (iv) Show that N must be an even number greater than 2. [8]

(b) It is often useful to consider the Dirac equation in dimensions other than four. Following the steps above, demonstrate that the Dirac equation in $D = 2+1$ dimensions is given by

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{\mathrm{i}}\left(\sigma_x\frac{\partial\psi}{\partial x} + \sigma_y\frac{\partial\psi}{\partial y}\right) + \sigma_z mc^2\psi\,,$$

where σ_x, σ_y, σ_z are Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ obeying $\sigma_i\sigma_k = \delta_{ik} + \mathrm{i}\epsilon_{ikl}\sigma_l$.

In $D = 3 + 1$ dimensions, consider the spin and angular momentum operators, whose components, correspondingly, are given by $S_i = -\mathrm{i}\varepsilon_{ijk}\alpha_j\alpha_k/4$ and $L_i = \mathrm{i}\varepsilon_{ijk}x_j p_k$, and introduce $J_i = L_i + S_i$.

Compute the commutators $[S_i, S_j]$, $[L_i, L_j]$ and $[J_i, J_j]$.

Show that $[H_D, J_i] = 0$. What are the quantum numbers associated with the Dirac wavefunction? [7]

(c) The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right) - mc\right]\psi = 0\,, \qquad (*)$$

where $\psi = \mathrm{e}^{-\mathrm{i}mc^2t/\hbar}\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $\hat{p}_\mu = \mathrm{i}\hbar\partial_\mu$, and the Dirac matrices are $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$. An alternative equation is proposed,

$$\left[\gamma^\mu\hat{p}_\mu - mc - \frac{i\kappa}{2c}F_{\mu\nu}\gamma^\mu\gamma^\nu\right]\psi = 0\,, \qquad (**)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor related to the electric and magnetic fields via $E_i = cF_{0i}$ and $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$, respectively. For equations (*) and (**), write down the system of coupled equations obeyed by the spinors $\tilde{\varphi}$ and $\tilde{\chi}$. [4]

(d) Find the equations obeyed by the spinor $\tilde{\varphi}$ to leading order in the non-relativistic limit of the equations (*) and (**). Compare the two results. What is the physical meaning of the parameter κ? [6]

Section S19 PARTICLE ACCELERATOR SCIENCE

1. SESAME (Synchrotron-light for Experimental Science and Applications in the Middle East) is a third-generation light source. The SESAME team may in future consider its upgrade by the installation of an on-energy injector booster, that will make injection nearly on-orbit, to allow the reduction of the apertures of all beamline elements.

(a) Assuming that the basic arc structure of SESAME is FODO, and that the upgrade will simply double the number of FODO cells per unit length, and that the wigglers and undulators do not significantly contribute to the beam emittance, estimate the expected reduction of the horizontal emittance. [9]

(b) Assuming that improved beamline alignment and improved field quality of the magnets will allow a reduction in the coupling between the horizontal and vertical planes from $k = 0.1$ to $k = 0.02$, estimate the expected reduction of the vertical emittance. [9]

(c) Taking the above into account, and assuming that the upgraded ring will still be far from diffraction-limited performance, estimate the expected increase of the brightness of the upgraded SESAME.

[*Hint: assume that the contribution of dispersion into the beam size is negligible.*] [7]

2. The future Electron Ion Collider (EIC) will utilize electron cooling to reduce the emittance of its proton beams. To enable electron cooling, the electron beam with CW current $I_e = 1\,\mathrm{A}$ is produced by a magnetized electron gun with thermionic cathode temperature $T_c = 1160\,\mathrm{K}$ (recall that $11600\,\mathrm{K} \approx 1\,\mathrm{eV}$). The parallel magnetic field, surrounding the gun, follows the electron beam throughout its acceleration to energy $E_e = 50\,\mathrm{keV}$. This electron beam then serves for the electron cooling of a proton beam circulating in a storage ring of the EIC.

(a) Explain why the electric power needed to produce this electron beam can be around two orders of magnitude lower than $1\,\mathrm{A} \times 50\,\mathrm{kV} = 50\,\mathrm{kW}$.

[*Hint: refer to the concept of energy recovery.*] [3]

(b) Give two other examples of energy recovery from various other areas of science and technology. [2]

(c) Explain why the equilibrium temperature of the cooled proton beam can be many orders of magnitude lower than T_c.

[*Hint: first show that the longitudinal temperature of the accelerated electron beam in the accompanying rest frame is given by* $T_{\|} \approx T_c^2/(4E_\mathrm{e})$.] [8]

(d) Estimate the equilibrium temperature of the cooled proton beam. [5]

(e) Explain why for magnetized electron beam cooling the transverse temperature of the accelerated electron beam is $T_\perp = T_c$ and why this does not affect the equilibrium temperature of the cooled proton beam. [7]

3. The International Linear Collider (ILC) project is aimed at colliding electron and positron beams with Center of Mass (CM) energy of 250 GeV, for detailed study of the Higgs boson. Correspondingly, the ILC consists of two linacs pointing to each other where the e^- and e^+ beams are accelerated to $E_0 = 125\,\mathrm{GeV}$ energy. The linacs operate at an RF frequency of $f_0 = 1.3\,\mathrm{GHz}$ and have average accelerating gradient of $G_{\mathrm{acc}} = 32\,\mathrm{MeV\,m^{-1}}$ (i.e., the length of each linac is 3.9 km).

It has been suggested to upgrade the ILC from a "Higgs factory", in the first instance, to a "top factory", to study top quark physics at a CM energy of 350 GeV, and to perform this upgrade by plasma acceleration using auxiliary proton beams.

The auxiliary proton beams will be accelerated in the first half of a re-phased ILC linac, to a central energy of $E_p = E_0/2$ while an energy chirp of $\Delta E/E$ will be created along the proton bunch of length ℓ_{p}. The accelerated proton bunch is then extracted from the linac and left to propagate over the long distance S_{prop} where the proton bunch is ballistically compressed to almost zero bunch length, for efficient plasma excitation. The electron bunches are accelerated in the second half of the linac to the energy $E_0/2$. The compressed proton bunches are brought just in front of the e^- and e^+ bunches, at the end of the corresponding linacs, to perform the energy boost using plasma acceleration cells.

(a) Assuming that $E_{\mathrm{p}} = 62.5\,\mathrm{GeV}$, $\ell_{\mathrm{p}} = 10\,\mathrm{mm}$ and $S_{\mathrm{prop}} = 1.95\,\mathrm{km}$, estimate the necessary energy chirp $\Delta E/E$ in the proton bunch. [7]

(b) Estimate the optimal transformer ratio k_T of plasma acceleration, which ensures the most efficient energy transfer from proton bunch to e^- and e^+ bunches. [5]

(c) What is the necessary ratio of the bunch population of the proton bunch to e^- or e^+ bunches $N_{\mathrm{p}}/N_{\mathrm{e}}$ in this case? [3]

(d) Estimate the plasma density n_0 assuming that the length of the plasma cell is $L = 1.1\,\mathrm{m}$. [5]

(e) What is the maximum CM energy of ILC upgraded by this method, taking into account that the transformer ratio k_T can be as high as six? [5]

Section S26 STARS AND GALAXIES

1. (a) Derive the equation of hydrostatic equilibrium within a star, at radius r enclosing mass M_r, and hence estimate the central temperature and pressure of the Sun, stating any assumptions you make. [7]

(b) Starting from the equation of hydrostatic equilibrium for a star, derive a relation between the total thermal energy, U, the total gravitational energy, Ω, and the ratio of heat capacities, $\gamma = C_P/C_V$.

If a star is composed of fully-ionized gas, what is the value of γ, and what is the resulting relation between Ω and U for the star? Use this to explain why nuclear burning in most stars is self-regulating, and also to explain the pre-main-sequence evolution of a gas cloud into a star.

Discuss briefly why γ may vary with position within a star. What will happen if $\gamma \leq 4/3$ in the outer layers of a star? [11]

(c) Radiation pressure is given by the expression

$$P_{\rm rad} = \frac{aT^4}{3}$$

where $a = 4\sigma/c$ is the radiation constant, and σ is the Stefan–Boltzmann constant. Use this to justify that the Sun's outer layers are supported against collapse primarily by gas pressure, $P_{\rm gas}$, not radiation pressure, $P_{\rm rad}$.

Derive a relation for how the ratio $P_{\rm rad}/P_{\rm gas}$ scales with stellar mass, and hence estimate the mass of stars which are supported against collapse mainly by radiation pressure. What limits the maximum mass of a star? [7]

2. (a) What is meant by *opacity* in astrophysics? At high temperatures, what source of opacity dominates? [4]

(b) Starting with the equations of stellar structure, justify that the luminosity of a star of mass M_s and central temperature T_c will vary approximately as

$$L \propto \kappa_0^{-1}\, \mu^7\, M_s^5\, T_c^{0.5}$$

[9]

[*You may assume that energy is transported solely by radiation, resulting in a temperature gradient*

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho L}{64\pi\sigma r^2 T^3}$$

where σ is the Stefan–Boltzmann constant. You may also assume that the star behaves as an ideal gas of mean molecular mass μ, and the opacity is given by Kramer's law: $\kappa = \kappa_0 \rho T^{-3.5}$, where κ_0 varies with chemical composition: $\kappa_0 \propto Z(1+X)$ where X and Z are the mass fractions of hydrogen and of elements heavier than helium, respectively.]

(c) Estimate the relative luminosities of the two stars described below:

(i) a star with $M_s = 1\, M_\odot$, central temperature $T_c = 1.5 \times 10^7\,\mathrm{K}$, composed mostly of hydrogen but with a small mass fraction Z of elements heavier than helium.

(ii) a star with $M_s = 0.5\, M_\odot$, central temperature $T_c = 10^8\,\mathrm{K}$, composed mostly of helium but with the same small mass fraction Z of elements heavier than helium as in star (i). [5]

(d) What is the observational evidence for the existence of stars similar to star (ii)? Outline the likely previous history of such a star. [7]

3. The Harvard classification sequence for stars has categories O, B, A, F, G, K, M (arranged in order of effective temperature from hot to cool).

(a) Outline the trends in spectral features seen across the categories. How can the absorption line strengths in the spectra of different stars vary so greatly, despite most stars having very similar chemical composition? Which stars have the strongest atomic hydrogen Balmer absorption lines (transitions from the $n = 2$ level)? Explain why this is the case, with reference to the effective temperatures of the stars. [7]

(b) The spectra of K-stars exhibit a strong triplet of absorption lines associated with singly-ionized calcium (Ca II), with the central line having wavelength 854.4 nm. An elliptical galaxy is observed to have a spectrum very similar to that of a K-star. What can you say about the stellar populations of this elliptical galaxy? If the central line of the Ca II triplet has an observed wavelength of 874.0 nm in the spectrum of the elliptical galaxy, calculate the redshift. Assuming a Hubble constant of $71\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, what is the distance of this elliptical galaxy? [6]

(c) Explain why the absorption lines of the Ca II triplet are noticeably broader in wavelength for the spectrum of the elliptical galaxy than for the spectrum of a single K-star. The spectral profile of the central absorption line in the elliptical galaxy is well fit by a Gaussian of dispersion $\sigma = 0.4\,\mathrm{nm}$, and the angular radius which contains half the light is 5 arc seconds. Estimate the total mass of this elliptical galaxy, stating any assumptions you make. If the integrated apparent magnitude within this radius is $m_V = 15.7$ magnitudes, calculate the total luminosity of this galaxy. What is the mass-to-luminosity ratio for this galaxy? Express your answer in solar units ($M_\odot/L_\odot$), and comment on your result. [8]

[*In V-band the absolute magnitude of the Sun is $M_V = 4.7$ magnitudes.*]

(d) What surface brightness profile do elliptical galaxies typically exhibit? And what is the scaling relation between luminosity and velocity dispersion for elliptical galaxies? [4]

Section S29 EXPLORING SOLAR SYSTEMS

1. Qualitatively explain what is meant by the term *hydrostatic equilibrium* in the context of a planet's atmosphere or interior. Write down (do not prove) the hydrostatic equation relating pressure, density and gravity. [3]

If the interior of a planet is assumed to be in hydrostatic equilibrium, is adiabatic and chemically homogenous, derive the Adams–Williamson equation for the internal density as a function of radius

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{-GM\rho(r)^2}{K_m r^2}.$$

You are given the definition of bulk modulus,

$$K_m = \rho \frac{\mathrm{d}P}{\mathrm{d}\rho},$$

where P is the pressure, and should state any further assumptions required. [5]

The Adams–Williamson equation can be used with seismic data to infer information on a planets internal density structure. If the P and S wave velocities are given by

$$V_\mathrm{P} = \sqrt{\frac{K_m + \frac{4}{3}\mu}{\rho}}, \qquad V_\mathrm{S} = \sqrt{\frac{\mu}{\rho}},$$

show that the Adams–Williamson equation can be re-written as

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM\rho(r)}{r^2\Phi}$$

where Φ is the seismic parameter defined in terms of V_P and V_S. What is the physical significance of the parameter μ and how does this relate to the differences in transmission between P and S waves? [5]

What are the typical limitations of this type of model? Name a common type of Solar System body that (with a few notable exceptions) is not in global hydrostatic equilibrium and explain your reasoning. [4]

To provide experimental constraints for any internal planetary model other measurements, in addition to seismic data, that remotely infer the interior structure of a planet are required. List two other possible techniques and the type of measurement data required to achieve this. [4]

Other than the Earth, as of June 2018, the only other solid planetary body to be investigated by seismology is the Earth's Moon using the instruments installed by the Apollo astronauts. With the help of a labelled diagram describe our current understanding of the interior of the Moon from the surface inwards. Name one significant limitation of the Apollo seismic network. [4]

2. Outline the key pieces of evidence that early Mars once had a significantly thicker atmosphere than we observe today. Describe some of the main processes over Mars's history that are likely to have led to the current climate we see on the planet today. [8]

How do we date the ages of planetary surfaces in the Solar System? Does this support the idea that different regions of the surface of Mars have very different ages? [2]

For impact driven atmospheric escape show that the mass of atmosphere lost during impact is given by

$$M_{\mathrm{Esc}} = \frac{\pi R_{\mathrm{imp}}^2 M_{\mathrm{atm}} V_{\mathrm{imp}}^2 R_{\mathrm{Mars}}}{2GM_{\mathrm{Mars}}},$$

where M_{Esc} is the mass of the atmosphere lost due to impact, R_{imp} is the radius of the impactor, V_{imp} is the velocity of the impactor, M_{atm} is the mass of the atmosphere per unit area, and R_{Mars} and M_{Mars} are the radius and mass of Mars respectively. Explain clearly any assumptions you make in the derivation. [4]

If early Mars had a CO_2 atmosphere with a surface pressure of approximately 1 bar, estimate,using the data below or otherwise, the fraction of the atmosphere that would be lost due to an impactor of diameter 300 km, approximately the scale of a possible asteroid that formed the Hellas basin on Mars. [8]

[$M_{\mathrm{Mars}} = 6.39 \times 10^{23}\,\mathrm{kg}$, $R_{\mathrm{Mars}} = 3.39 \times 10^{6}\,\mathrm{m}$, $V_{\mathrm{imp}} = 15\,\mathrm{km\,s^{-1}}$.]

Does this support the theory that Mars could have lost the bulk of its atmosphere by impact driven escape? What other evidence from the current atmospheric composition of Mars and the morphology of planetary surfaces in the inner Solar System might also support this theory? [3]

3. Outline qualitatively the important stages involved in the Core Nucleation Theory of giant planet formation. What are the main pieces of evidence that support the theory and what are its limitations? [10]

A very simplified version of part of this theory can be used to provide a first order estimate of the timescales for the initial phase of a giant planet's formation. After the planet has accreted the bulk of its mass it radiates away its kinetic energy content over a characteristic Kelvin–Helmholtz timescale given by

$$t_{\rm KH} \approx \frac{GM^2}{R_{\rm proto}L},$$

where G is the gravitational constant, M is the mass of the proto-planet, $R_{\rm proto}$ is the radius of the proto-planet and L is the planet's intrinsic luminosity.

Explain what is meant by the term *luminosity* for a planet and show that if we assume the proto-planet has a configuration similar to the current state of Jupiter then the luminosity is roughly $3.42 \times 10^{17}\,\mathrm{W}$. [7]

To estimate $t_{\rm KH}$ requires knowledge of the radius of the proto-planet's *Hill sphere*, given by

$$R_H = r_p\left(\frac{M}{3M_{\rm Sun}}\right)^{1/3}.$$

Why is this distance scale important for understanding the rate of formation of the planet? Using this information, or otherwise, estimate the formation time scale for a Jupiter sized planet at 5.2 AU from the Sun, assuming Solar luminosity, planetary composition etc. are the same as we find in the current Solar System. Explain whether your estimate is compatible with the key assumptions of the Core Nucleation theory for giant planet formation. [8]

[*Data you may find useful: equatorial radius of Jupiter* $= 6.9911 \times 10^7\,\mathrm{m}$*; distance from Sun = 5.2 AU; Bond albedo = 0.343, energy balance (emitted/absorbed power) = 1.67.*]